THE

MATHEMATICAL

GAZETTE

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc

AND

Prof. E. T. WHITTAKER, M.A., F.R.S.

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PROBABILITY AND STATISTICS.

By W. F. Sheppard, Sc.D., LL.M.

Statistical method is coming to be so extensively applied, that those who use it must from time to time feel the need of assuring themselves as to the security of its foundations. For this assurance they will naturally turn to books on the subject. But do these give much help?

A recent book, which has received much laudation, is Mr. J. M. Keynes' *Treatise on Probability*. With the main portion, comprising Parts I.-IV., I am not concerned; it has been very fully considered by Mr. Bertrand Russell in a recent notice.* But Part V. is headed "The Foundations of Statistical Inference," and the student of statistics will naturally turn to it for guidance. This part Mr. Russell does not profess to appraise: he merely says that it "contains much admirable criticism." A reviewer in a monthly journal † says, apparently with reference to this part:

> "The student of statistics will be well advised to read what the author has to say about the subject of statistical inference";

and this may perhaps be taken as typical of notices of the book.

As Part V. occupies 100 8vo pages—roughly 36,000 words—and cannot be fully understood without some reference to the earlier portions of the book, the perusal of it is a rather formidable task; and the student of statistics may well ask, "What benefit shall I get from it?" My own impression is that it will, according to the extent of his knowledge, either mislead or exasperate him.

One difficulty which confronts the statistical reader, if he has been trained in the exact use of words, is the difficulty of knowing what Mr. Keynes means. On p. 328, for example, he speaks of employing a frequency curve, obtained from data as to deaths at different ages, "to determine the probability of death at a given age in the population at large." What does "probability" mean here? Is it the actuary's $(l_x - l_{x+1})/l_x$, which is an estimated statistical ratio? Or is it the probability, in the old sense, of the death of an individual? Or is it a measure of rational belief of something, and, if so, of what? The most reasonable explanation is that "the probability of death at a given age" is simply a mistake for "the corresponding frequency."

* *Mathematical Gazette*, vol. xi., July 1922, pp. 119-125.

† *Discovery*, vol. iii., October 1922, p. 278.

But a still more serious fault of the book is its irrelevance. The theory of rational belief is one thing: the mathematical theory of probability—which is practically dead, except as a mathematical exercise and a matter of historical and psychological interest, its place (and part of its nomenclature) having been taken by the frequency-calculus—is another: and the science of statistics—an inductive science, based or to be based on observation and experiment, and using the frequency-calculus as its deduction-machine—is another. This science is being built up slowly: and Mr. Keynes does not give much help by putting up fragments of the old theory of probability and knocking them down again.

Chapter XXX., for instance, deals with the "rule of succession." But how many actuaries, let us say—for actuaries form a class to whom statistical inference is of special importance—have ever heard of this "rule," or, if they did hear of it, would think it worthy of the slightest attention? It is true that Mr. Keynes is able to quote a good many names in support of the rule: this is interesting historically—it would have been more interesting if dates had been given—but it does not make the rule a foundation-stone.

Mr. Keynes, in the third paragraph of his concluding chapter, which professes to be constructive and should perhaps therefore be regarded as one of the most important in the book, says (p. 406):

> "A statistical induction either asserts the probability of . . . or else it assigns the probability of . . ."

This seems to mean that the function of statistical induction is to determine probability. The writer of the review referred to above makes the converse statement that:

> ". . . the frequency theory of probability . . . is the foundation of most statistical work."

I doubt if any large number of statisticians would assent to either of these statements. It is quite true that statisticians do use such phrases as "probable error," which seem to connote probability. But probability, if the word is used in the ordinary sense, is one thing, frequency is another. You cannot infer frequency from probability, nor probability from frequency; unless, in either case, probability means frequency, which is practically what it does mean to the statistician.

This is, I take it, the fundamental difference between statisticians (when they really consider what they are doing) and Mr. Keynes. To quote the latter again (pp. 384-5, italics mine):

> "If we knew that our material could be likened to a game of chance, we might expect to infer chances from frequencies, *with the same sort of confidence as that with which we infer frequencies from chances.*"

The reference is obviously to Bernoulli's theorem; and I may perhaps be allowed to make some detailed observations on Mr. Keynes' treatment of this theorem.

The problem, as stated on p. 330, is:

> "Given the probability relative to certain evidence of each of a series of events, what are the probabilities, relative to the same evidence, of various proportionate frequencies of occurrence for the events over the whole series? Or more briefly, how often may we expect an event to happen over a series of occasions, given its probability on each occasion?"

The problem is considered in Chapter XXIX., where it is stated in the following form (p. 337):

> "Given a series of occasions, the probability of the occurrence of a certain event at each of which is known relative to certain initial *data h*, on what proportion of these occasions may we reasonably anticipate the occurrence of the event? Given, that is to say, the individual prob-

ability of each of a series of events *à priori*, what statistical frequency of occurrence of these events is to be anticipated over the whole series ? "

Now one of the difficulties of reading this book is the constant looseness of phrase. What I think the above passages mean is :

> "Given, for each of a series of occasions of the possible happening of an event, the probability that it will actually happen, what is the probability that in the series as a whole the frequency [*i.e.* the proportionate number of times] of its happening will have a stated value ?"

The solution of the problem is given by Bernoulli's theorem. The theorem is in two parts. Briefly it is as follows. Let the probability that when the occasion A happens the event B will happen be always p. Suppose that A happens m times, and that X is the proportion of cases in which B happens. Then (1) the value of X which has the greatest probability is p (or $1/m$ of the integer nearest to mp), and (2) the probability that X will lie between $p-\xi$ and $p+\xi$ is a certain integral which, when ξ is fixed, increases as m increases, and may be made as nearly equal to 1 as we like by taking m large enough.

In reference to this theorem Mr. Keynes says (p. 341), "it will have to be conceded that it is only true of a special class of cases." This will puzzle the reader, for the reasoning seems quite sound ; but the next sentence shows that by *true* Mr. Keynes means *applicable*, which is quite a different thing—the example he gives of a case in which it is not applicable being a case which does not fall within the mathematical theory of probability at all, since the multiplication-property does not hold.

This, apparently, is the theorem as stated by Bernoulli himself. But what has this to do with the inferring of frequencies from chances ? The answer seems to lie in the further deduction, which, according to Venn (*Logic of Chance*, 3rd ed., p. 91), "is generally expressed somewhat as follows :—in the long run all events will tend to occur with a relative frequency proportional to their objective probabilities." (I cannot find in Venn the statement, attributed to him by Mr. Keynes on p. 341, that if the series is "long enough" the proportion will *certainly* be p.) But again Mr. Keynes is so loose in his enunciation of propositions that I cannot make out whether he accepts this deduction or not. He says on p. 333 that Bernoulli showed that, if the *à priori* probability is known throughout, then (subject to certain conditions) *in the long run* a certain determinate frequency of occurrence "is to be expected" ; on p. 341, "We seem . . . to have proved . . . that if the series is a long one the proportion is very unlikely to differ widely from p" ; on p. 344, that the theorem supplies a formula by which we can "attempt" to pass from *à priori* probabilities to a prediction of statistical frequency. These are all rather vague. But on p. 361 he says that the theorem "predicts" what is, on certain evidence [*i.e.*, no doubt, the evidence to which the probability is relative], "likely" to happen. On p. 367 he goes still further : he speaks of "the complementary part of our enquiry,—namely, into the method of deriving a measure of probability from an observed statistical frequency," which can hardly do otherwise than imply that what we have previously done was to derive a frequency from a probability. And perhaps the most decisive passage is on p. 352, where he treats a probability ·9999779 as "practical certainty."

My own explanation of this indecisiveness or obscurity is that Mr. Keynes has not thought out the matter : that he has, to put it in its simplest form, accepted without question the view that to state that the probability of an event is 1 is the same thing as to state that it will certainly happen. But this is just the sort of thing that he ought to have examined carefully : for it contains a big fallacy.

In the mathematical theory of probability, or, as it is better called, the calculus of probability, events have probabilities assigned to them according

to a certain system, one feature of which is that if it is known that an event will happen its probability is taken to be 1, if it is known that it will not happen its probability is taken to be 0, and for other cases the probability is taken to be between 0 and 1. The fallacy lies in inverting this, and saying that if the deduced probability of a certain event is 1 the event will certainly happen, or that if it is 0 the event will certainly not happen. The fallacy must have been pointed out many times, but I cannot find any mention of it in Mr. Keynes' book.

If frequency cannot be inferred from probability, nor probability from frequency, a good deal of Mr. Keynes' reasoning falls to the ground.

In another respect Mr. Keynes does not seem to understand the statistician. He speaks sometimes as if the object of collecting statistics were to infer more statistics. Thus he begins Part V. (p. 327) by saying that the theory of statistics can be divided into two parts, its first function being descriptive and its second inductive: and he explains this last word by saying that the theory seeks to extend its description of certain characteristics of observed events to the corresponding characteristics of other events which have not been observed. But this is a very incomplete account. It is true that in some cases a statistical ratio, such as the proportion of persons of a stated age who survive for a year, is observed for a limited aggregate with a view to forming an estimate for a larger aggregate. But this is only a particular kind of case, and indeed a very elementary case, of application of the science. The scientific student collects facts, and arranges them statistically: from this arrangement, as Mr. Keynes correctly says (p. 327), he deduces quantitative descriptions of salient characteristics of the data; and he usually proceeds to consider what can be stated with regard to a hypothetical "population," from which the data may be regarded as obtained by a process of random sampling. But this last step is a purely conventional and intermediate one. The salient characteristics—dispersion, correlation, and so on—are the result of causes, some of which are essential, while the remainder, including those which confer the quality of randomness, are accidental. The object of the scientific statistician is to determine the essential or long-run causes.

To take a trivial example, Mr. Keynes, at the end of the chapter on Bernoulli's theorem, quotes certain statistics as to dice-throwing, roulette, and the like, and concludes with the statement (p. 366):

> ". . . it is worth emphasising once more that from such experiments as these this is the only *kind* of knowledge which we can hope to obtain,—knowledge of the material construction of a die or of the psychology of a croupier."

But this is just the sort of knowledge for the sake of which such statistics would be collected. An ordinary die is necessarily unsymmetrical, more material being cut out of it for the larger numbers. It may therefore be expected that in the long run a six will come most frequently, a five next, and so on. The theory is tested, and is found to be correct; and the figures give material for an estimate as to the proportion of times that a particular number or combination or sequence of numbers will be thrown with the particular kind of die. The matter is trivial in itself, but the method and result of the experiment are extremely significant.

In distinguishing between the descriptive and the inductive stages, Mr. Keynes is rather worried about a supposed liability of statisticians to confuse the two. He says (p. 327):

> "The statistician, who is mainly interested in the technical methods of his science, is less concerned to discover the precise conditions in which a description can be legitimately extended by induction. He slips somewhat easily from one to the other, and having found a complete and satisfactory mode of description he may take less pains over the transitional

argument, which is to permit him to use this description for the purposes of generalisation."

And again (p. 329):

> "... the more complicated and technical the preliminary statistical investigations become, the more prone inquirers are to mistake the statistical description for an inductive generalisation.... The term 'probable error'... is used *both* for the purpose of supplementing and improving a statistical description, *and* for the purpose of indicating the precision of some generalisation. The term 'correlation' itself is used *both* to describe an observed characteristic of particular phenomena *and* in the enunciation of an inductive law which relates to phenomena in general."

Statistical investigators are quite aware of the necessity for paying attention to mistakes of observation (such as those due to "personal equation"), errors of observation, and errors of randomness; and it would have been interesting to have cases quoted in which matters of such obvious importance had been neglected in any work accepted as being of authority. W. F. SHEPPARD.

GLEANINGS FAR AND NEAR.

190. *Divisible* is applied to numbers contained an integral number of times in the dividend—*e.g.* 12 is divisible by 6 and not by 8; or "divisible without producing fractions in result...."

In making complicated divisions, it is much the shortest plan, and very much the safest, to begin by forming the first nine multiples of the divisor by continued addition (forming the tenth for proof).—De Morgan, *P.C.*, "Division."

191. Si quis mathematicus fuerit, id est, invocator daemonum....

Martene et Duran, *Vet. Script et Mon. amplissima Collectio*, Tom. 7, § 33.

192. ... the necessity of Latin and Mathematics, the one soon forgot, and the other never got to any purpose.—Walpole to Mann, May 24, 1760.

193. The word impossible has been so misused in algebra, in the sense of inexplicable, that the impossible of ordinary life, which "can't be—and never, never comes to pass," requires some additional epithet to express it in an algebraical work. (Footnote to "... it means that it is really and truly impossible that....")—De Morgan's *Trig. and Double Alg.* 1849, p. 112.

194. Before going up to Cambridge the whim took Thackeray to consult a phrenologist named Deville, who kept a lamp shop at 367 Strand, and in the intervals of legitimate business felt the bumps of clients in the warehouse over his shop. Deville was either French or Swiss, but had lived long enough in London to become quite a cockney in his talk. He was an Athenian in his love for novelties, and was as enthusiastic about mesmerism as about phrenology. He was a man of wide reading, and it was said that you might talk to him on any subject and learn something from him. Dr. Elliotson, who was detested by his medical brethren for his advocacy of the merits of mesmerism, said that he was in the same position as Deville with regard to detractors—"they are conwinced but won't awow it." Deville was a man of discretion. To one of his customers he remarked, after feeling his bumps: "You have a love of approbation." Then said the client, "You would call me a vain man." "No, sir, I would not call it wanity, but love of distinction." Whatever his verdict was in the case of Thackeray, Deville said to him as he was leaving the warehouse: "Come back, and I will tell from your bumps whether you have been reading classics or mathematics." It matters little what Deville's verdict was, for Thackeray retorted: "Sold! I haven't opened a page of either."

THE POSTULATE OF PARALLELS.

By Prof. M. J. M. Hill, F.R.S.

I desire to suggest to those of your readers, who are giving instruction in the Elements of Geometry, that they should try an experiment when they arrive at the stage in which it is necessary to introduce the Postulate of Parallels.

Whether this be given in Euclid's or in Playfair's form it must come as a shock to the thoughtful reader. There is nothing in his previous experience to prepare him for it. He may well ask: "Why should I believe this?"

It seems to me that here, just as in the case of the renowned Fifth Definition of the Fifth Book of the *Elements* (the Test for the Equality of Two Ratios), the sub-structure, by which the definition was originally reached, has been cut away.

What was that sub-structure?

Most probably it was obtained, at any rate in the first instance, by taking a point O outside a straight line a, and then considering the position assumed by a straight line joining O to a point P moving along a to a great distance first in one direction and then in the other, and observing that the further P moved in either direction the more nearly did OP tend to take up a single definite position, viz.: that of a straight line b, which was such that a and b were cut by any transversal so as to make the sum of the two interior angles on the same side together equal to two right angles. Generalizing then *beyond the limits of possible experience*, the postulate of parallels was reached.

On the other hand, it might have been obtained by a method *based upon experience*, viz.: from the idea, first, I believe, stated by J. Wallis in the seventeenth century in the following terms: "*To every figure there exists a similar figure of arbitrary magnitude.*" This idea occurred later to Carnot (1803) and to Laplace (1824) (see Bonola's *Non-Euclidean Geometry*, pp. 53-4). Laplace regards it as a more natural postulate than that of Euclid. On the other hand, Bonola says (*l.c.* pp. 16-17): "But even though intuition would support this view, the idea of form, independent of the dimensions of the figure, constitutes a hypothesis, which is certainly not more evident than the Postulate of Euclid."

The idea would appear to be innate. As soon as a child can recognize an object from a picture the idea has been acquired. In an experiment, which I once made with a picture of a watch, a child nine months old, on seeing the picture, said, "tic, tic," thus showing that the object had been recognized. Possibly, perhaps I should say probably, the idea is in fact acquired at an even earlier age.

I think, therefore, it can properly be used as a point of departure.

1. Let us apply the idea to a triangle.

The assumption is that any number of triangles of arbitrary magnitude can be constructed having the same shape as a given one.

If two triangles have the same shape, then the angles of the one triangle are respectively equal to those of the other triangle.

Let ABC be a given triangle.

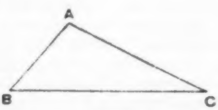

Fig. 1.

Take any length $B'C'$. Draw through B' the straight line $B'X$, making

$X\hat{B}'C' = A\hat{B}C$. Draw through C' the straight line $C'Y$, on the same side of $B'C'$ as $B'X$, and in the same plane as $B'C'$ and $B'X$, and making $Y\hat{C}'B' = A\hat{C}B$.

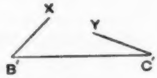

FIG. 2.

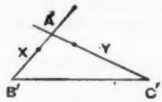

FIG. 3.

Then the assumption that a triangle can be constructed of the same shape as ABC, but of any size, involves the fact that $B'X$ and $C'Y$ must meet in some point A', and that $B'\hat{A}'C' = B\hat{A}C$ (Fig. 3).

2. In what follows this construction will be made by taking C' at C and B' anywhere on BC, then $B'X$ makes with BC the angle $X\hat{B}'C = A\hat{B}C$, whilst $C'Y$ coincides with CA and A' is the point where $B'X$ meets CA.

This may be put shortly thus:

If ABC be a triangle, and if any point D be taken on BC (or BC produced), and if through D a straight line be drawn so as to make with DC the same

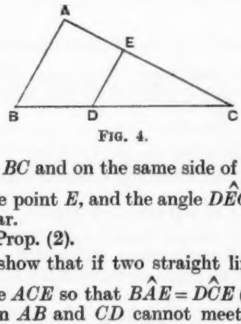

FIG. 4.

angle as AB makes with BC and on the same side of BC, then it will meet AC (or AC produced) in some point E, and the angle $D\hat{E}C = B\hat{A}C$ and the triangles DEC, BAC will be similar.

I will refer to this as Prop. (2).

3. The next step is to show that if two straight lines AB, CD in the same plane meet a straight line ACE so that $B\hat{A}E = D\hat{C}E$ (AB and CD being on the same side of ACE), then AB and CD cannot meet. (Wallis does not give this application of his postulate. It will be found in a paper by Professor Nunn on p. 71 of the *Mathematical Gazette* for May, 1922.)

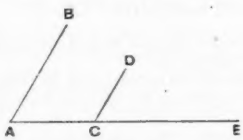

FIG. 5.

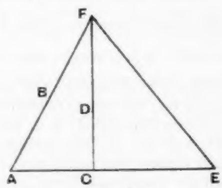

FIG. 6.

If possible let AB, CD meet at some point F. Join EF.

Now AEF is a triangle (Fig. 6).

Let us apply Prop. (2) to it and construct a triangle of the same shape as AEF, but with CE corresponding to AE. Then we have to draw through C a straight line making with CE an angle equal to $B\hat{A}E$, this is by hypo-

thesis CD; and through E a straight line making with EC an angle equal to AEF, this is EF. The third vertex of this triangle is F, and we should, if Wallis's postulate is admitted, have $C\hat{F}E = A\hat{F}E$, which is impossible. Hence CD cannot meet AB, if $B\hat{A}E = D\hat{C}E$.

4. The next step is to show that if $B\hat{A}E = D\hat{C}E$, then every straight line through C, other than CD, must meet AB.

On CD take any point G, and join GE.

Then GCE is a triangle (Fig. 7).

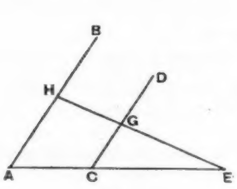

FIG. 7.

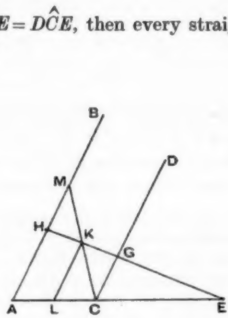

FIG. 8.

Then, by Prop. (2), if we take A on CE, and draw through A a straight line AB making $B\hat{A}E = G\hat{C}E$, it must meet GE in some point H, and $A\hat{H}E = C\hat{G}E$.

Now straight lines through C, other than CD, must lie either in the angle $A\hat{C}G$ or in the angle $E\hat{C}G$ (Fig. 8).

Consider first those which lie in the angle $A\hat{C}G$. Any such straight line if prolonged far enough must cross the boundary made up by the lines AH and HG.

If it cross AH between A and H it meets AB.

If, however, it cross HG between H and G, let it cross at the point K.

Now AHE is a triangle, K is a point on HE, and if through K a straight line be drawn making with KE an angle equal to AHE it must by Prop. (2) meet AE in some point L.

Moreover, the line KL must, if prolonged far enough, cross the boundary made up of the lines HA, AC and CG. But it cannot meet HA, since $E\hat{K}L = E\hat{H}A$, and it cannot meet CG, because $E\hat{K}L = E\hat{G}C$. Hence it must cross AC, and therefore L lies between A and C. Further $K\hat{L}E = H\hat{A}E$.

Now KLC is a triangle, and A is a point on LC, and $C\hat{A}H = C\hat{L}K$.

Therefore, by Prop. (2), CK and AH meet in some point M and $A\hat{M}C = L\hat{K}C$.

Thus, if CK crosses HG between H and G it meets AB.

If the line through C is in the angle ECG, all that is necessary is to produce BA through A, DC through C, then take G anywhere on the production of DC, and repeat the construction.

[Wallis's proof (see Bonola, *l.c.* pp. 15-16) involves the consideration of the *continuous* motion of a straight line in such a manner that it makes a constant angle with a fixed straight line.

Professor Nunn, *l.c.* pp. 71-2, bases his proof of Euclid's postulate on several propositions, one of which is the proposition that if there are three angles $\hat{B}$, $\hat{C}$ and $\hat{R}$ whose sum is equal to two right angles, then it is possible to construct a triangle whose angles are respectively $\hat{B}$, $\hat{C}$ and $\hat{R}$; but Professor

Nunn does not give the method by which he gets this proposition out of the postulate of similarity (Wallis's postulate). He has since sent me a full statement of his argument. The proof set out in this article obtains the required result without first obtaining from the postulate of similarity the propositions which Professor Nunn uses.]

5. The result of §§ 3 and 4 is that through any point C one and only one straight line can be drawn in the plane CAB, which does not meet the straight line AB (which does not pass through C). This is Playfair's form of the Postulate of Parallels.

6. It follows from § 4 that if through A and C two straight lines AB and CK be drawn, such that $B\hat{A}C + A\hat{C}K = G\hat{C}E + A\hat{C}K <$ two right angles, then AB and CK must meet.

This is Euclid's form of the Postulate of Parallels.

The experiment I suggest is that those who have to teach the Elements of Geometry should adopt some such method as I have given above to introduce the Postulate of Parallels.

Would it not be possible to carry the great majority of minds which are brought to a full stop by the bare enunciation of the Postulate of Parallels through the reasoning set forth above to an intelligent appreciation of the Postulate?

University College, London, M. J. M. Hill.
7th September, 1923.

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195. Besant's *Notes on Roulettes and Glissettes*, 1870, probably the first or only mathematical book ever noticed in *Punch*. Of this Besant was justly proud, for the work was his own, even to the word "Glissette," which he invented, and he had the cutting from *Punch* pasted into his own copy:—

Punch, 2nd April, 1870.

What can they be?

Notes on Roulettes and Glissettes.—This sounds rather frisky for the title of a book by a "Lecturer and Fellow of St. John's College, Cambridge," and we should feel easier as to the future of St. John's College, if we could receive an assurance from the College Authorities that they have examined the work in question, and can vouch for its being of the highest respectability.

196. The word ORTHOCENTRE was invented by Besant and Ferrers while walking on the Trumpington Road about the year 1865; and, after consulting many classical scholars, Besant endeavoured to substitute PHORONOMY for "Kinematics," but the latter was too firmly established to be changed.—*Proc. Lond. Math. Soc.* Ser. ii. vol. 16, part 5.

197. . . . We are so accustomed to contradictory combinations, used in some emphatic sense, that they are recognised idioms: it even happens that they express more and better meaning while they are fresh, and before use makes the contradiction wear off, than afterwards. When General Wolfe first used the expression "choice of difficulties," which was contradiction, choice then meaning voluntary election, he made those to whom he wrote see his position with much more effect than could have been produced a second time by the same words. Ordinary language has methods of instantaneously assigning meaning to contradictory phrases: and thus it has stronger analogies with an algebra (if there were such a thing) in which there are pre-organised rules for explaining new contradictory symbols as they arise, than with one in which a single instance of them demands an immediate revision of the whole dictionary. . . .—De Morgan's *Trig. and Double Alg.* 1849 (footnote), p. 90.

FURTHER NOTES ON APPROXIMATE INTEGRATION.

By Arnold Buxton, M.A.

In paragraphs III. and IV. of my recent paper on Approximate Integration, general formulae were obtained to include all empirical rules with equidistant and chosen ordinates respectively. Both these theorems are, however, particular cases of a more general theorem, which is not so widely known as it deserves.

To obtain a general theorem including all empirical rules for both equidistant and chosen ordinates.

The problem is again the determination of the area shown in Fig. 1 between the curve $y=F(x)$, the axis of x, and two given ordinates $x=p$ and $x=q$.

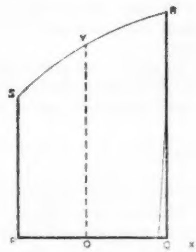

Fig. 1.

Let $$y=a_0+a_1x+a_2x^2+\dots+a_{2s-1}x^{2s-1}. \qquad \text{.......................(1)}$$

represent a curve having $2s$ points in common with $y=F(x)$, which is assumed to be smooth over the region considered, and having practically the same area as that between (1), the ordinates $x=p$, $x=q$ and the axis of x.

The required area

$$=\int_p^q (a_0+a_1x+a_2x^2+\dots+a_{2s-1}x^{2s-1})\,dx$$

$$=a_0(q-p)+a_1(q^2-p^2)/2+a_2(q^3-p^3)/3+\dots+a_{2s-1}(q^{2s}-p^{2s})/2s. \qquad \text{.....(2)}$$

The determination of previous empirical rules in the paper already referred to has shown that (2) can be written as $\sum_{r=1}^{r=s_1} B_r y_r$, where $y_1, y_2, \dots y_{s_1}$ represent ordinates of (1), and $B_1, B_2, \dots B_{s_1}$ are the corresponding coefficients or weighting factors of those ordinates. Therefore we have the identity,

$$a_0(q-p)+a_1(q^2-p^2)/2+a_2(q^3-p^3)/3+\dots+a_{2s-1}(q^{2s}-p^{2s})/2s$$
$$\equiv B_1y_1+B_2y_2+\dots+B_{s_1}y_{s_1}.$$

Now let $x_1, x_2, \dots x_{s_1}$ be the abscissae corresponding to the ordinates $y_1, y_2, \dots y_{s_1}$.

Then
$$\begin{aligned}\text{R.H.S.} &= B_1(a_0+a_1x_1+a_2x_1^2+\dots+a_{2s-1}x_1^{2s-1})\\ &+B_2(a_0+a_1x_2+a_2x_2^2+\dots+a_{2s-1}x_2^{2s-1})\\ &+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &+B_{s_1}(a_0+a_1x_{s_1}+a_2x_{s_1}^2+\dots+a_{2s-1}x^{2s-1}).\end{aligned}$$

Further, let the origin be taken at the mid-point of PQ, and let the length of PQ be 2 units; then $q=1$, $p=-1$, and the identity becomes

$$2[a_0+a_2/3+a_4/5+\ldots+a_{2s-2}/(2s-1)]$$
$$=a_0\sum_1^{s_1}B_r+a_1\sum_1^{s_1}B_rx_r+\ldots+a_{2s-1}\sum_1^{s_1}B_rx_r^{2s-1}.$$

Comparing coefficients of $a_0, a_1, a_2, \ldots a_{2s-1}$ on both sides,

$$\left.\begin{array}{ll}\sum_1^{s_1}B_r\quad=2, & \sum_1^{s_1}B_rx_r^4\quad=\frac{2}{5},\\ \sum_1^{s_1}B_rx_r\ =0, & \ldots\ldots\ldots\ldots\ldots\ldots\\ \sum_1^{s_1}B_rx_r^2=\frac{2}{3}, & \sum_1^{s_1}B_rx_r^{2s-2}=\dfrac{2}{2s-1},\\ \sum_1^{s_1}B_rx_r^3=0, & \sum_1^{s_1}B_rx_r^{2s-1}=0.\end{array}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The expression (3) consists of $2s$ equations involving not more than $2s_1$ unknowns. In the case of equidistant ordinates, the abscissae $x_1, x_2, \ldots x_{2s}$ are known, for the approximation curve (1) is defined by $2s$ equidistant ordinates with abscissae

$$-1,\quad -1+2/(2s-1),\quad -1+4/(2s-1),\quad \ldots\quad -1+2(2s-1)/(2s-1).$$

Hence the quantities to be determined from (3) are the $2s$ coefficients, $B_1, B_2, \ldots B_{2s}$ and $s_1=2s$.

If $s_1=s$, then the coefficients $B_1, B_2, \ldots B_s$, and the abscissae of the ordinates $x_1, x_2, \ldots. x_s$ have to be determined by means of the equations (3), which now become

$$\sum_1^{s}B_r=2,\qquad \sum_1^{s}B_rx_r=0,\quad \text{etc.}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3\text{A})$$

Now the Gaussian rules were based on the theorem that s chosen ordinates can be found to give the same accuracy as $(2s-1)$ equidistant ordinates. Hence, we are now provided with an alternative proof to this theorem, for the solutions of (3A) must give the same approximation as those obtained by solving equations (3) for equidistant ordinates. Incidentally, we notice that a convenient method of solving (3A) is, firstly, to take the abscissae as the roots of the Legendre Function $P_s(x)=0$, and, secondly, substituting these values in the equations to solve for the coefficients. If $s_1<s$, then the number of abscissae or ordinates is less than s and, as the number of coefficients must equal the number of ordinates (one to each ordinate), it follows that the sum of ordinates and coefficients is less than $2s$. Consequently, the equations (3) will not, in general, be satisfied. *Therefore it follows that the number of coefficients and ordinates must be equal to* $2s$. It is to be noted that the Gaussian rules employ the least number of ordinates.

The utility of this theorem, however, occurs when, instead of choosing the ordinates first and then determining the coefficients, the coefficients are made to follow a simple law and the abscissae of the ordinates evaluated afterwards. Tchebycheff has investigated the most useful case, viz. when the coefficients are all equal, *i.e.* let

$$B_1=B_2=B_3=\ldots B_{s_1}.$$

Then the abscissae to be determined will be $x_1, x_2, x_3, \ldots x_{s_1}$ (in other words, the unknowns are s_1 abscissae and one coefficient), *i.e.* $s_1+1=2s$ or $s_1=2s-1$.

Hence equations (3) become

$$\left.\begin{array}{l}(2s-1)B_1=2,\quad B_1\sum_1^{2s-1}x_r=0,\quad B_1\sum_1^{2s-1}x_r^2=2/3,\ \ldots,\\ B_1\sum_1^{2s-1}x_r^{2s-2}=2/(2s-1),\quad B_1\sum_1^{2s-1}x_r^{2s-1}=0,\end{array}\right\}\quad\ldots\ldots\ldots\ldots\ldots(4)$$

giving $2s$ equations to determine $x_1, x_2, \ldots, x_{2s-1}, B_1$. More generally let the coefficients be constant multiples of B_1, say, $c_1B_1, c_2B_1, c_3B_1, \ldots, c_{2-1}B_1$. Then equations (4) will be modified as follows:

$$B_1 = 2/\Sigma c, \quad \sum_1^{2s-1} c_r x_r = 0, \quad \sum_1^{2s-1} x_r^2 = 2c/3, \quad \ldots, \quad \sum_1^{2s-1} c_r x_r^{2s-2} = \Sigma c/(2s-1), \quad \sum_1^{2s-1} x_r^{2s-1} = 0.$$

An elementary sequence of c's such as 1, 2, 3, 4, 3, 2, 1 should lead to an interesting empirical rule, although, for convenient use, it is difficult to improve upon the Tchebycheff system, where it is merely a question of adding up the ordinates and multiplying by B_1.

II. *The evaluation of simple cases from the general formula.*

(i) Consider an example of equidistant ordinates by taking Simpson's Rule. $y = a_0 + a_1x + a_2x^2$ is the approximation curve, and the equations (3) now become:

$$B_1 + B_2 + B_3 = 2, \quad B_1x_1 + B_2x_2 + B_3x_3 = 0, \quad B_1x_1^2 + B_2x_2^2 + B_3x_3^2 = 2/3.$$

The *three* equidistant ordinates will have as abscissae

$$-1, \quad -1 + \frac{2}{2}, \quad -1 + \frac{2(2)}{2}, \quad i.e. \ -1, \ 0, \ 1.$$

Hence $\quad B_1 + B_2 + B_3 = 2, \quad -B_1 + B_3 = 0, \quad B_1 + B_3 = 2/3,$

$$i.e. \ B_1 = B_3 = 1/3, \quad B_2 = 4/3.$$

(ii) Next consider a simple case of the Gaussian Rule.

Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ be the equation of the approximation curve. The number of equidistant ordinates will be 6 (including first and last), and the number of the Gaussian chosen ordinates will be 3.

The equations (3A) are now

$$B_1 + B_2 + B_3 = 2, \quad B_1x_1 + B_2x_2 + B_3x_3 = 0, \quad B_1x_1^2 + B_2x_2^2 + B_3x_3^2 = 2/3,$$
$$B_1x_1^3 + B_2x_2^3 + B_3x_3^3 = 0, \quad B_1x_1^4 + B_2x_2^4 + B_3x_3^4 = 2/5, \quad B_1x_1^5 + B_2x_2^5 + B_3x_3^5 = 0.$$

The abscissae are given by $P_3(x) = 0$,

$$i.e. \quad x_1 = -\sqrt{3}/\sqrt{5}, \quad x_2 = 0, \quad x_3 = \sqrt{3}/\sqrt{5}$$

(see Dale's *Mathematical Tables*, under the heading Zonal Harmonics),

$$i.e. \quad B_1 + B_2 + B_3 = 2, \quad B_1 = B_3.$$

$$\left.\begin{aligned} B_1 3/5 + B_3 3/5 &= 2/3, \\ B_1 9/25 + B_3 9/25 &= 2/5, \end{aligned}\right\} \quad i.e. \quad \begin{aligned} B_1 &= B_3 = 5/9, \\ B_2 &= 8/9. \end{aligned}$$

Hence the rule, taking PQ as 2 units of length,

$$\text{Area} = 5/9\,[(\text{ordinate at } x = -\sqrt{3}/\sqrt{5}) + (\text{ordinate at } x = \sqrt{3}/\sqrt{5})]$$
$$+ 8/9\,[(\text{ordinate at } x = 0)].$$

(iii) Next, consider a simple case of the Tchebycheff Rules.

Again taking the approximation curve,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5,$$

we have, in the Tchebycheff system, to evaluate $B_1, y_1, y_2, y_3, y_4, y_5$.

Equations (4) become

$$5B_1 = 2. \quad \ldots\ldots\ldots(1) \qquad B_1\sum_1^5 x_r^3 = 0. \quad \ldots\ldots\ldots(4)$$

$$B_1\sum_1^5 x_r = 0. \quad \ldots\ldots\ldots(2) \qquad B_1\sum_1^5 x_r^4 = 2/5. \quad \ldots\ldots\ldots(5)$$

$$B_1\sum_1^5 x_r^2 = 2/3. \quad \ldots\ldots\ldots(3) \qquad B_1\sum_1^5 x^5 = 0. \quad \ldots\ldots\ldots(6)$$

(2), (4) and (6) are satisfied by $x_1 = -x_5,\quad x_2 = -x_4,\quad x_3 = 0.$

Hence $B_1 = 2/5,\quad 2(x_1^2 + x_2^2) = 5/3,\quad 2(x_1^4 + x_2^4) = 1,$

i.e. $x_1^2 = (5 + \sqrt{11})/12,\quad x_2^2 = (5 - \sqrt{11})/12,\quad x_3 = 0.$

$x_1 = \cdot 8325 = -x_5,\quad x_2 = \cdot 3745 = -x_4,\quad x_3 = 0.$

Hence the rule for obtaining the area *PQRS* in Fig. 2. Denote the mid-ordinate at M (the mid-point of PQ) by y_3. Measure it and the ordinates on each side of it distant $\pm\cdot 3745MQ$ (y_2 and y_4 respectively) and $\pm\cdot 8325MQ$ (y_1 and y_5 respectively) from y_3. Area $= \frac{2}{5}(y_1 + y_2 + y_3 + y_4 + y_5)$.

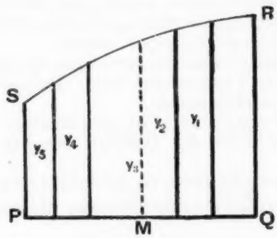

FIG. 2.

Similar rules can be obtained by increasing or decreasing the number of ordinates. Briefly considering the case of 9 ordinates, these with B_1 involve 10 unknowns, and so the approximation curve is of the 9th degree.

$$B_1 = 2/9,\quad B_1\Sigma x = 0,\quad B_1\Sigma x^2 = 2/3,\quad B_1\Sigma x^3 = 0,\quad \ldots,\quad B_1\Sigma x^8 = 2/3,\quad B_1\Sigma x^9 = 0.$$

The 2nd, 4th, 6th, 8th, and 10th equations give

$$x_1 = -x_9,\quad x_2 = -x_8,\quad x_3 = -x_7,\quad x_4 = -x_6,\quad x_5 = 0.$$

Thus $\Sigma x^2 = 3,$ *i.e.* $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 3/2,$

$\Sigma x^4 = 9/5,\qquad x_1^4 + x_2^4 + x_3^4 + x_4^4 = 9/10,$

$\Sigma x^6 = 9/7,\qquad x_1^6 + x_2^6 + x_3^6 + x_4^6 = 9/14,$

$\Sigma x^8 = 9/9,\qquad x_1^8 + x_2^8 + x_3^8 + x_4^8 = 1/2.$

It can be shown that $x_1^2, x_2^2, x_3^2, x_4^2$ are the roots of the quartic

$$x^4 - 1\cdot 5x^3 + \cdot 675x^2 - \cdot 1023x + \cdot 0025 = 0.$$

Forming from this the equation whose roots are the squares of those of the quartic, and repeating the process until the first two terms of the quartic with roots of the 8th power of those of the original quartic are obtained, the largest root is now isolated, giving $x_1^2 = \cdot 831$.

Substituting this value of x_1^2 for $x_1^2, x_1^4, x_1^6, x_1^8$ in the above equations, the cubic of which x_2^2, x_3^2, x_4^2 are the roots can be formed and solved by the ordinary methods, giving

$$x_2^2 = \cdot 361,\quad x_3^2 = \cdot 280,\quad x_4^2 = \cdot 0282,$$

so that Area $= \frac{2}{9}(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)$,

where	y_1, y_9	are ordinates	corresponding	to	abscissae	$\pm\cdot 912$,
	y_2, y_8	,,	,,	,,	,,	$\pm\cdot 601$,
	y_3, y_7	,,	,,	,,	,,	$\pm\cdot 529$,
	y_4, y_6	,,	,,	,,	,,	$\pm\cdot 168$,
	y_5	,,	,,	,,	,,	0.

A. BUXTON.

ON SOME PROPERTIES OF THE EPIPEDON.

By Prof. E. L. Watkin, M.A.

The solid which is the three-dimensional analogue of the plane quadrilateral appears to have been neglected by geometers, possibly on account of the difficulty of visualising it adequately from a plane figure. A model, however, makes an interesting object and is readily constructed from thin stiff wire and thread. The extraordinary number of collinearities, which are its principal feature, then become at once obvious. For the benefit of any one who wishes to construct such a model I would suggest the desirability of commencing with the external diagonal triangle PQR and the line $P'Q'R'$ (see Figs. 1 and 2), as otherwise it is difficult to get a compact model.

The solid does not appear to have received any special name, and for want of an earlier or better one I suggest the term "Epipedon" on account of its obvious relation to the parallelepiped.

The following brief account may be of some interest.

Calling the vertices $ABCDA'B'C'D'OPQR$ (see Fig. 1), we see that it is made up of:

12 *points* lying by *threes* on 16 *straight lines*,

12 *planes* passing by *threes* through 16 *straight lines*.

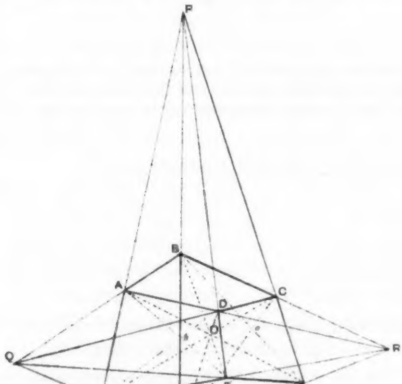

Fig. 1.

Through each *point* pass 6 *planes* and 4 *lines*.

In each *plane* lie 6 *points* and 4 *lines* (forming a complete quadrilateral).

Each *line* contains *three points* and lies in *three planes*.

Any plane which contains only three of the points may be regarded as a *diagonal plane* of the epipedon, *e.g.* PQR, and starting with any point at random of a collinear triad, three other points can at once be found such that none of their joins contains a fourth of the points, and these four points, such as $OPQR$, determine four diagonal planes forming a *diagonal tetrahedron*.

There are therefore *three diagonal tetrahedra*, viz. $OPQR$, $AB'CD'$, $A'BC'D$ and *twelve diagonal planes*.

If o, o' be the internal diagonal points of the quadrilaterals $ADB'C'$ and $A'D'BC$, then o, O, o' are collinear in the meet of the planes $AA'BB'$ and $CC'DD'$, *i.e.* in OQ, and a similar statement is true for every other pair of corresponding diagonal points of a pair of "opposite" faces.

All the diagonal points of the plane quadrilaterals formed from the eight points $ABCDA'B'C'D'$ thus lie on the edges of the diagonal tetrahedron $OPQR$.

Section by a diagonal plane. Let AA', BB', CC', DD' meet the diagonal plane PQR in a, b, c, d (see Fig. 2).

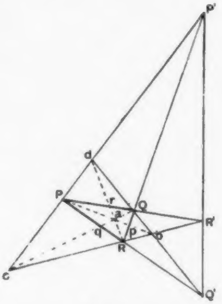

FIG. 2.

Then PQR is inscribed in, and in perspective with bcd, the centre of perspective being a. Let $P'Q'R'$ be the corresponding axis of perspective.

Similarly RPQ is inscr. in, and in persp. with cda, centre b, axis $P'qr$.

„ PQR „ „ „ dab, „ c, „ $pQ'r$.

„ RQP „ „ „ abc, „ d, „ pqR'.

Also BD, $B'D'$ meet at P'; CD', $C'D$ at Q'; BC', $B'C$ at R';

AC, $A'C'$ meet at p; AB', $A'B$ at q; AD', $A'D$ at r.

The planes PQR, $BC'D$, $B'CD'$ have thus a common axis $P'Q'R'$, and similarly three diagonal planes pass through each of the lines $P'qr$, $Q'rp$, $R'pq$. Also there are four of these axes in each of the faces of the tetrahedron $OPQR$. There are therefore 16 axes in all, which lie by fours in the 12 faces of the three diagonal tetrahedra, each of which contains six of the twelve diagonal points of the faces of the epipedon (each such point being a diagonal point for three faces), and these points lie by threes on the 16 axes.

Hence *the twelve diagonal points of the faces of the epipedon form a complete epipedon themselves, and the relation between the two epipeda is reciprocal, the diagonal points of the faces of each being the vertices of the other and the faces of one the diagonal planes of the other.*

Properties of the mid-points of diagonals. Since the mid-points of the diagonals of a plane quadrilateral are collinear, we have the following collinearities:

The mid-points of AB', $C'D$, PR,

„ „ $A'B$, CD', PR,

„ „ AB', $A'B$, OQ,

„ „ CD', $C'D$, OQ (see Fig. 3).

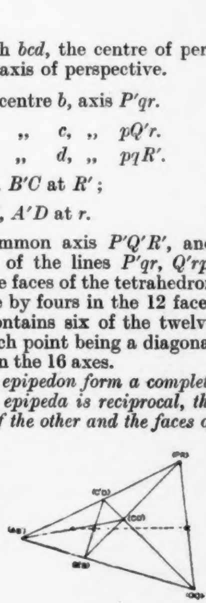

FIG. 3.

The mid-points of these six lines are therefore coplanar, and lie by threes on four straight lines, forming a complete quadrilateral.

The mid-points of the edges of the diagonal tetrahedra are therefore collinear by threes on 12 straight lines, and coplanar by sixes in three planes.

The mid-points of the diagonals of each of the new quadrilaterals so formed are clearly the centroids of the diagonal tetrahedra, hence the three planes just obtained are coaxial and *the centroids of the diagonal tetrahedra are collinear.*

Hyperboloids of one sheet associated with the epipedon. Consider the hyperboloid of one sheet (H_1) determined by the lines BC, DB', $C'D'$.

All these lines are met by CD, $B'C'$, $D'B$, which are therefore generators of the opposite system, and the twisted hexagon $BCDB'C'D'$ lies wholly on the surface.

The H_1 passes through P, Q, R and touches one of the planes of the figure at each of the nine points $BCDB'C'D'PQR$.

Hence, omitting each collinear triad in turn, we see that there are *sixteen H_1's, each containing nine of the vertices, and six of the edges of the epipedon and touching nine of its planes.*

Nature of the intersection of two such H_1's. Consider first the case where the two omitted triads have a point in common, *e.g.* AOA', BOB'.

The two hyperboloids $BB'CC'DD'PQR$, $AA'CC'DD'PQR$ have the intersecting generators CDQ and $C'D'Q'$ in common, hence their remaining intersection is a conic.

But each of these H_1's passes through P and R, and their tangent planes at these points intersect in OP and OR, and therefore their common conic touches OP and OR at P and R.

A second conic touching OP and OR at P and R is obtained from the H_1's determined by omitting COC' and DOD'.

There are three diagonal tetrahedra, each face of which furnishes six such conics.

Hence there are in all 72 *conics which are the partial intersections of these hyperboloids, nine lying on each hyperboloid.*

The nine conics on each H_1 are arranged in three groups of three. Each of a group touches the other two, and their planes pass through one or other of the vertices which do not lie on the H_1.

If the omitted triads have no point in common, *e.g.* AOA', PBD', the H_1's are at once found to have two generators of the same system in common, QCD, $RB'C'$. They have therefore *two other non-intersecting generators in common,* viz. the two lines which meet AA', BD', CD, $B'C'$.

Clearly *four of the H_1's pass through each of these lines,* since three lines determine an H_1. Also each pair of lines counts as six intersections. There are eight such groups, giving 16 lines counting as 48 intersections, and these with the 72 conics make up the 120 intersections of the hyperboloids.

It will readily be found that if the three points omitted are not collinear, the hyperboloids through the remaining nine points degenerate into a pair of planes.

Thus *with every epipedon there are associated sixteen hyperboloids of one sheet passing through nine of the vertices and six of the edges, and touching nine of the faces. These intersect in pairs either in two intersecting edges and a conic touching two diagonals, or in two non-intersecting edges and two other non-intersecting lines, which are common to four of the hyperboloids.*

University College, Southampton. E. L. WATKIN.

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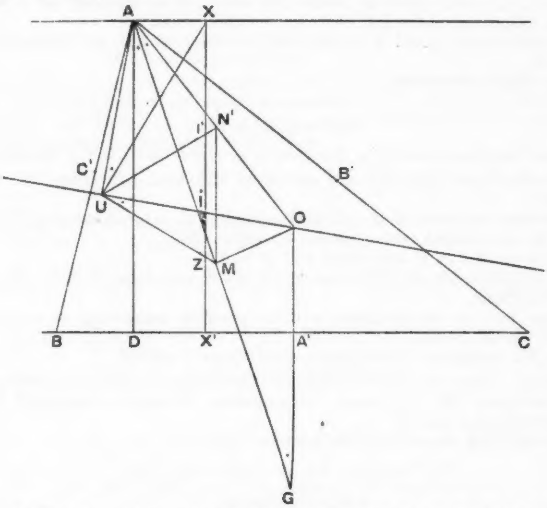
198. In the time of Wakefield it was a matter of common observation that the inhabitants of the northern counties "are usually the profoundest proficients in mathematics and philosophy."—*Life of Gilbert Wakefield*, 1804, i. pp. 83, 84.

.MATHEMATICAL NOTES.

692. [K¹. 2. d.] *Feuerbach's Theorem.*

ABC is a triangle, OI the common diameter of the circum- and in-circles (radii R, r). AD, XIX', OG are lines perpendicular to BC, G lying on the circumcircle, so AIG is a straight line. Draw AU perpendicular to OI and OM perpendicular to AIG. Join MU cutting XI at Z. [The lines AG, XX', OU should concur.]



Then, if N' is the middle point of AO and UN' cuts XI at I', $N'M$ is parallel to OG and $\therefore$ to XIX'; also $N'M=N'U$; $\therefore$ $I'Z=I'U$.

Again, $\angle OUM=\angle OAM=\angle AIX=\angle AUX$, since A, U, I, X lie on a circle.

$$\therefore \angle XUZ=\angle AUO=\text{a right angle.}$$

Hence I' is the centre of the circle XUZ; so the circles on XZ and AO as diameters touch at U.

Now the triangles AUX, OGI are equiangular $\rightarrow \begin{cases}\angle AXU=\angle AIU=\angle OIG,\\ \angle AUX=\angle AIX=\angle OGI.\end{cases}$

Also the right-angled triangles AUI, XUZ are equiangular; $\angle UAI=\angle UXI$;

$$\therefore \frac{IG}{OG}=\frac{UX}{UA}=\frac{XZ}{AI}.$$

Hence $$OG\,.\,XZ=AI\,.\,IG=2Rr;$$

$$\therefore XZ=2r.$$

Now reflect these two circles in the line $B'C'$, joining the mid-points of AC, AB.

The circle on XZ as diameter becomes the in-circle; the circle on AO as diameter becomes the 9-point circle.

Hence the 9-point circle touches the in-circle at the point ω, which is the image of U in the line $B'C'$.

Clifton, 31st January, 1922. E. P. LEWIS.

693. [K^1. 8.] *On a Certain Quadrilateral.*

A rectangle or parallelogram, with its sides in the ratio of 2 : 1, has this property that the four bisectors of its angles meet two and two on opposite sides (namely, the longer pair of sides).

Does a more general type of quadrilateral possess this property?

A. *Geometrical Investigation.* Let AB, one side of the quadrilateral $ABML$ and the two angles at A and B, be given.

Call these given elements p, α, β respectively.

Then q, λ, μ will serve to name the side LM and angles at L and M. And the other two sides LA, MB can be called x and y.

The bisectors of A and B are supposed to meet on LM, and those of L and M on AB.

Of the eight quantities,

$$\text{sides } p,\ q,\ x,\ y,$$
$$\text{angles } \alpha,\ \beta,\ \lambda,\ \mu,$$

it is clear that the given p, α, β suffice to provide formulae for the other five.

1. *Construction.* AL, BM are given in situation. Produce LA, MB to meet in C.

Bisect the angles at C, A and B (internal at C, external at A and B) so as to find Q, an escribed centre for the triangle ABC.

Let CQ cut AB in P, and draw $PD \perp^r$ to CA.

With P as centre describe the circle BDF, and from Q draw $MQFL$ to touch this circle.

(Either two or no solutions will be possible, according to a condition presently to be given.)

Then ML completes the required quadrilateral $ABML$.

2. *Proof.* Since the circle touches the three sides of CML it is the inscribed circle, and thus PM, PL bisect the angles at M and L. And QB, QA are the bisectors of A and B.

To be possible, the radius PD must be $<PQ$.

Now $$PQ/PA = \sin PAQ/\sin PQA = \sin\frac{\alpha}{2}\Big/\cos\frac{\beta}{2}$$

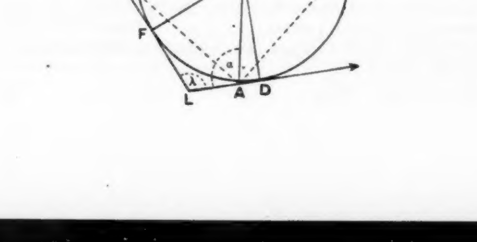
and $$PD/PA = \sin\alpha\,;$$

$$\therefore\ \sin\alpha < \sin\frac{\alpha}{2}\Big/\cos\frac{\beta}{2} \quad\text{or}\quad 2\cos\frac{\alpha}{2}\cos\frac{\beta}{2} < 1.$$

This is the necessary preliminary condition.

When this holds good two quadrilaterals, crossed or uncrossed, will be possible.

3. If we take for given quantities the side LA and the angles λ, α, we can



get QP (by bisecting angles A, L); produce it to meet LD in C and make $Q\hat{C}M = Q\hat{C}L$, which gives MB.

This always gives a possible construction.

Or, let PL, AQ cut in K, and describe the circle PQK; then the bisector of the angle made by QL, PA produced will cut the circle in K, and again in a point L; and QLB, PLM can be drawn to determine the line MB.

4. We shall now give expressions, in terms of p, α, β, for
the other sides and angles of $ABML$,
the area of $ABML$,
the symmetrical line PQ,
and other quantities,

observing that since the fundamental geometrical property of the quadrilateral is symmetrical, such quantities as the above, expressed in terms of p, α, β, will also have equivalent expressions in terms of q, λ μ and hence many trigonometrical identities will result.

5. $$\sin\lambda\sin\mu = \cos\left(\frac{\alpha-\beta}{2}\right)\left(3\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right).$$

6. Thus μ, λ, the other two angles of the quadrilateral, are connected by an equation with α, β, and as $\alpha+\beta+\lambda+\mu = 2\pi$, we find that

$$\lambda = \pi - \frac{\alpha+\beta}{2} \pm \cos^{-1} 2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}, \quad \mu = \pi - \frac{\alpha+\beta}{2} \mp \cos^{-1} 2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}.$$

7. $$PQ = 2p\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\Big/(\sin\alpha+\sin\beta) = 2q\sin\frac{\lambda}{2}\sin\frac{\mu}{2}\Big/(\sin\lambda+\sin\mu);$$

$$\therefore\ q\sin\lambda\sin\mu = p\sin\alpha\sin\beta,$$

or $$q = p\sin\alpha\sin\beta/\sin\lambda\sin\mu$$

$$= p\sin\alpha\sin\beta/\cos\frac{\alpha-\beta}{2}\left(3\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right), \text{ by (5)}.$$

8. $$LA = x = p\sin\left(\alpha+\frac{\lambda}{2}\right)\sin\beta/2\sin\frac{\alpha+\beta}{2}\sin\lambda\cos\frac{\mu}{2},$$

$$MB = y = p\sin\left(\beta+\frac{\mu}{2}\right)\sin\alpha/2\sin\frac{\alpha+\beta}{2}\sin\mu\cos\frac{\lambda}{2},$$

which alone do not readily lend themselves to explicit expression in terms of p, α, β.

9. The ratio x/y is thus either

$$= \sin\left(\alpha+\frac{\lambda}{2}\right)\sin\beta\sin\frac{\mu}{2}\Big/\sin\left(\beta+\frac{\mu}{2}\right)\sin\alpha\sin\frac{\lambda}{2},$$

or by a symmetrical interchange of α with λ, and β with μ,

$$= \sin\left(\lambda+\frac{\alpha}{2}\right)\sin\mu\sin\frac{\beta}{2}\Big/\sin\left(\mu+\frac{\beta}{2}\right)\sin\lambda\sin\frac{\alpha}{2};$$

whence $$\left(\cot\alpha+\cot\frac{\lambda}{2}\right)\Big/\left(\cot\lambda+\cot\frac{\alpha}{2}\right) = \left(\cot\beta+\cot\frac{\mu}{2}\right)\Big/\left(\cot\mu+\cot\frac{\beta}{2}\right).$$

10. The area of the quadrilateral $ABML$

$$= \triangle CML - \triangle CBA = \tfrac{1}{2}p^2\sin\alpha\sin\beta\sin\frac{\alpha+\mu}{2}\sin\frac{\beta+\mu}{2}\Big/\sin\lambda\sin\mu.$$

Or again,

$$= p^2\frac{t_1t_2}{(1+t_1t_2)(t_1+t_2)}\,\frac{3+t_1t_2}{3-t_1t_2}, \quad \text{where } t_1 = \tan\frac{\alpha}{2},\ t_2 = \tan\frac{\beta}{2}.$$

11. Area $=\frac{1}{2}p^2\sin\alpha\sin\beta\sin\frac{\alpha+\mu}{2}\sin\frac{\beta+\mu}{2}=\frac{1}{2}q^2\sin\lambda\sin\mu\sin\frac{\lambda+\alpha}{2}\sin\frac{\mu+\alpha}{2}$

and $$p\sin\alpha\sin\beta=q\sin\lambda\sin\mu.$$

Hence $$\sin\alpha\sin\beta\sin\frac{\lambda+\alpha}{2}\sin\frac{\mu+\alpha}{2}=\sin\lambda\sin\mu\sin\frac{\alpha+\mu}{2}\sin\frac{\beta+\mu}{2}.$$

12. All these results can be proved by Trigonometry (as in next section B), or tested in either of these two special cases:

(i) put $p=2q$ and $\alpha=\beta=\lambda=\mu=90°$;

(ii) put $\alpha=\beta=\lambda=120°$, $\mu=0$;

p finite, $MA=LM=\infty$, and $BL=0$.

B. *Trigonometrical.* Without any reference to a quadrilateral, we can construct a purely trigonometrical argument, arriving at the same results, as follows.

The basic theorem is this:

1. If $\alpha+\beta+\lambda+\mu=2\pi$, and if, in addition, $2\cos\frac{\lambda}{2}\cos\frac{\mu}{2}=\cos\frac{\alpha-\beta}{2}$,

then, reciprocally, $$2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}=\cos\frac{\lambda-\mu}{2}.$$

Numerous symmetrical, or partly symmetrical results come out as follows (the order of statement keeping close to the order of proof):

2. $$2\sin\frac{\lambda}{2}\sin\frac{\mu}{2}=3\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2},$$

$$\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\cos\frac{\lambda}{2}\cos\frac{\mu}{2}=\cos\frac{\beta+\mu}{2}\cos\frac{\alpha+\mu}{2},$$

$$\sin\lambda\sin\mu=\cos\frac{\alpha-\beta}{2}\left(3\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)$$

$$=\cos\frac{\alpha-\beta}{2}\left(2\cos\frac{\alpha+\beta}{2}+\cos\frac{\alpha-\beta}{2}\right).$$

3. $$4\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\lambda}{2}\cot\frac{\mu}{2}=\left(\cot\frac{\alpha}{2}\cot\frac{\beta}{2}+1\right)\left(\cot\frac{\lambda}{2}\cot\frac{\beta}{2}+1\right),$$

or $$3LA-L-A-1=0,\text{ where } L=\cot\frac{\lambda}{2}\cot\frac{\mu}{2},\quad A=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}.$$

Written otherwise, $2(1-LA)=(L-1)(A-1)$ or, clearing off the denominators of the cotangents,

$$2\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\lambda}{2}\cos\frac{\mu}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\lambda}{2}\sin\frac{\mu}{2}\right)$$

$$=-\cos\frac{\lambda+\mu}{2}\cos\frac{\alpha+\beta}{2}=\cos^2\frac{\alpha+\beta}{2}.$$

4. $$\cos\frac{\alpha-\beta}{2}+\cos\frac{\lambda-\mu}{2}=3\cos\frac{\lambda}{2}\cos\frac{\mu}{2}+\sin\frac{\lambda}{2}\sin\frac{\mu}{2},$$

and also is equal to any of the following:

$$2\sin\frac{\beta+\mu}{2}\sin\frac{\beta+\lambda}{2},\quad 2\sin\frac{\alpha+\mu}{2}\sin\frac{\alpha+\lambda}{2},$$

$$2\sin\frac{\lambda+\alpha}{2}\sin\frac{\lambda+\beta}{2},\quad 2\sin\frac{\mu+\alpha}{2}\sin\frac{\mu+\beta}{2}.$$

This symmetrical expression plays an important part; we may call it "*P*."

5. $$\cos(\alpha-\beta)-\cos(\lambda-\mu)=2\sin(\beta+\mu)\sin(\alpha+\mu)$$
$$=P\cos\frac{\lambda+\mu}{2}=\cos\lambda+\cos\mu-\cos\alpha-\cos\beta.$$
$$P=\{\cos(\alpha-\beta)-\cos(\lambda-\mu)\}\sec\frac{\lambda+\mu}{2}=(\cos\lambda+\cos\mu-\cos\alpha-\cos\beta)\sec\frac{\lambda+\mu}{2}.$$

6. $$\sin\alpha\sin\beta-\sin\lambda\sin\mu=\frac{1}{2}\{\cos(\alpha-\beta)-\cos(\lambda-\mu)\}$$
$$=P\left(\cos\frac{\alpha-\beta}{2}-2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\right)=P\cos\frac{\lambda+\mu}{2}=\cos\alpha\cos\beta-\cos\lambda\cos\mu.$$

7. $$\sin\alpha\sin\mu-\sin\beta\sin\lambda=\sin(\lambda+\alpha)\sin(\lambda+\mu).$$

8 $$\cos\lambda+\cos\mu=-4\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2},$$
$$\sin\lambda+\sin\mu=\sin\alpha+\sin\beta+\sin(\alpha+\beta).$$

9. $$\cos\alpha+\cos\beta+\cos\lambda+\cos\mu=4\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha+\lambda}{2}\cos\frac{\beta+\lambda}{2}$$
$$=-2\cos^2\frac{\alpha+\beta}{2}$$
$$=4\left(\Pi\sin\frac{\alpha}{2}-\Pi\cos\frac{\alpha}{2}\right),\quad\text{by (3)}.$$

10. $$\cos\frac{\alpha+\beta}{2}=2\cos\frac{\alpha+\lambda}{2}\cos\frac{\alpha+\mu}{2},\quad\cos\frac{\lambda+\mu}{2}=2\cos\frac{\lambda+\alpha}{2}\cos\frac{\lambda+\beta}{2}.$$

11. $$\sin\lambda+\sin\mu-\sin\alpha-\sin\beta=\sin\overline{\alpha+\beta}=-\sin(\lambda+\mu),$$
$$\cos\lambda+\cos\mu-\cos\alpha-\cos\beta=2P\cos\frac{\lambda+\mu}{2},\quad\text{see (4)},$$
$$=4\cos\frac{\lambda+\mu}{2}\sin\frac{\lambda+\alpha}{2}\sin\frac{\lambda+\beta}{2}$$
$$=2\sin(\lambda+\alpha)\sin(\lambda+\beta),\quad\text{by (10)}.$$

12. $$\cos\alpha-\cos\beta+\cos\lambda-\cos\mu=4\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\lambda}{2}\sin\frac{\mu+\alpha}{2}$$
$$=\sin(\alpha+\beta)\tan\frac{\mu+\alpha}{2},$$
$$\cos\alpha-\cos\beta-\cos\lambda+\cos\mu=\sin(\alpha+\beta)\tan\frac{\lambda+\alpha}{2},$$
$$\sin\alpha-\sin\beta+\sin\lambda-\sin\mu=4\cos\frac{\alpha+\beta}{2}\sin\frac{\beta+\mu}{2}\cos\frac{\beta+\lambda}{2}$$
$$=2\cos^2\frac{\alpha+\beta}{2}\tan\frac{\beta+\mu}{2},$$
$$\sin\alpha-\sin\beta-\sin\lambda+\sin\mu=4\cos\frac{\alpha+\beta}{2}\sin\frac{\beta+\lambda}{2}\cos\frac{\beta+\mu}{2}$$
$$=2\cos^2\frac{\alpha+\beta}{2}\tan\frac{\beta+\lambda}{2}.$$

13. $$\tan\frac{\lambda}{2}+\tan\frac{\mu}{2}=\frac{2\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\quad\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}=\frac{2\sin\frac{\lambda+\mu}{2}}{\cos\frac{\lambda-\mu}{2}}.$$

14.
$$\Sigma \tan\frac{a}{2} = 2P\sin\frac{\alpha+\beta}{2}\Big/\cos\frac{\alpha-\beta}{2}\cos\frac{\lambda-\mu}{2}$$
$$= 2\sin\frac{\alpha+\beta}{2}\left(\sec\frac{\alpha-\beta}{2}+\sec\frac{\lambda-\mu}{2}\right)$$
$$= \sin\frac{\alpha+\beta}{2}\sin\frac{\alpha+\lambda}{2}\sin\frac{\alpha+\mu}{2}\Big/\Pi\cos\frac{a}{2},$$
$$\Sigma\cot\frac{a}{2} = \sin\frac{\alpha+\beta}{2}\cos\frac{\mu+\alpha}{2}\cos\frac{\mu+\beta}{2}\Big/\Pi\sin\frac{a}{2}.$$

15.
$$\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\lambda}{2}\cos\frac{\mu}{2} = \sin\frac{\alpha}{2}\sin\frac{\beta}{2}+\sin\frac{\lambda}{2}\sin\frac{\mu}{2}$$
$$= \text{half sum} = \frac{1}{2}P = \sin\frac{\mu+\alpha}{2}\sin\frac{\mu+\beta}{2}.$$

16.
$$\frac{\sin(\alpha+\lambda)+\sin\alpha}{\sin(\alpha+\lambda)+\sin\lambda} = \frac{\sin(\beta+\mu)+\sin\beta}{\sin(\beta+\mu)+\sin\mu}$$
$$= \frac{\sin\alpha+\sin\beta}{\sin\lambda+\sin\mu}$$
$$= \frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\lambda-\mu}{2}} = \frac{\cos\frac{\lambda}{2}\cos\frac{\mu}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}.$$

17. $\sin(\alpha+\lambda)+\sin\alpha = 2\cos\frac{\lambda}{2}\cos\frac{\mu}{2}\left\{\sin\frac{\alpha+\beta}{2}-\sin\frac{\alpha-\beta}{2}-\sin\frac{\lambda-\mu}{2}\right\}$

(α interchangeable with β, and λ with μ)
$$= 2\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}\cdot\sin\frac{\alpha-\lambda}{2}\cos\frac{\alpha+\lambda}{2},$$

from which the results of the last section can be deduced.
$$\sin\left(\alpha+\frac{\lambda}{2}\right)\cos\frac{\beta}{2} = \sin\left(\lambda+\frac{\alpha}{2}\right)\cos\frac{\mu}{2},$$
$$\frac{\cot\alpha+\cot\frac{\lambda}{2}}{\cot\lambda+\cot\frac{\alpha}{2}} = \frac{\cot\frac{\lambda}{2}\cot\frac{\mu}{2}}{\cot\frac{\alpha}{2}\cot\frac{\beta}{2}} = \frac{\cot\beta+\cot\frac{\mu}{2}}{\cot\mu+\cot\frac{\beta}{2}},$$
$$\frac{\cos\alpha-\cos(\alpha+\lambda)}{\cos\lambda-\cos(\alpha+\lambda)} = \frac{\sin\frac{\lambda}{2}\cos\frac{\mu}{2}}{\sin\frac{\alpha}{2}\cos\frac{\beta}{2}}.$$

18. Each of the above can be tested by taking for the angles α, β, λ, μ the set of values

either	90°,	90°,	90°,	90°,
or	120°,	120°,	120°,	0°.

which satisfy the hypotheses
$$\alpha+\beta+\lambda+\mu = 2\pi \quad\text{and}\quad \cos\frac{\alpha-\beta}{2} = 2\cos\frac{\lambda}{2}\cos\frac{\mu}{2}.$$

G. E. Crawford.

PROBLEMS AND SOLUTIONS.

SOLUTIONS.

279 [J. 1. b.], I. p. 217. If N denotes the total number of lines in the figure and M the number of lines through each of the original points, the total number of points of intersection is obviously

$$\tfrac{1}{2}N(N-1)-mn\{\tfrac{1}{2}M(M-1)-1\},$$

assuming that the figure is plane and that there are no accidental coincidences. To find the number explicitly in terms of m and n, we must substitute

$$M=m(n-1)+1,\ N=\tfrac{1}{2}m^2n(n-1)+n,$$

but the result does not tidy up to the point of acquiring, to my mind, the faintest interest.

We may check the formula by a different line of argument. If A is one of the points on one of the given lines a, and B one of the points on another line b, then the M lines through A include a and AB, and the M lines through B include b and AB, and therefore of the $N-n$ cross-lines, there are $2M-3$ which pass through A or B. Hence the number of intersections *not* on the original lines is $\tfrac{1}{2}(N-n)(N-n-2M+3)$, and to find the total we have to add the original mn points points, the $\tfrac{1}{2}n(n-1)$ intersections of the given lines, and $\tfrac{1}{2}m^2n(n-1)(n-2)$ points where one of the original lines is cut by a line joining points on two other lines.

There might be some interest in going to the next stage and joining every pair of points now obtained, for there are Pascal collinearities in the figure if m is not smaller than 3; but I imagine the investigation for general values of m and n would be very hard. E. H. N.

326 [I. 2. b.; J. 2. a.], I. p. 280. Let the multiplier be one whose inherent probability of doing the work correctly (apart from test) is p. Then before the test is applied we have the probabilities:

	Stands test	Fails at test
Multiplication correct	p	0
" wrong	$(1-p)/9$	$8(1-p)/9$

Final probability of correct answer is $p/[p+(1-p)/9]=9p/(8p+1)$.

In other words: if the multiplier gets a wrong answer once in n times, then once in $9n-8$ times he will get a wrong answer which satisfies the 9-test.

The special case $n=1$ is of interest.

Eton College. W. Hope-Jones.

374 [L¹. 3. a.], I. p. 372. If (α, β) is the pole of the chord, the envelope equation to the parabola is

$$a^2X^2+b^2Y^2-1+(\alpha X+\beta Y+1)^2=0.$$

Since the axis is parallel to $x/\alpha=y/\beta$ it passes through the pole of

$$\alpha x+\beta y=\alpha^2+\beta^2,$$

which is $a^2\alpha X+b^2\beta Y+\alpha^2+\beta^2=0$; thus the equation to the axis is

$$x/\alpha-y/\beta=(a^2-b^2)/(\alpha^2+\beta^2).$$

In the special case of the normal chord, the equation becomes

$$x\cos\phi/a^3+y\sin\phi/b^3=(a^2-b^2)^2/(a^6\sec^2\phi+b^6\operatorname{cosec}^2\phi).$$

A. Robson.

REVIEWS.

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By E. W. Hobson. Second Edition. Volume I. 1921. (Cambridge University Press.)

Professor Hobson's treatise on *The Theory of Functions of a Real Variable*, published by the Cambridge University Press sixteen years ago, was at the time the only systematic account of theories so novel in their character, even to the ordinary professional mathematician, that author and publishers alike may have well had doubts as to the success of the venture. The appearance after an interval of less than fifteen years of the first of the two volumes which are to constitute the second edition is, especially when the monumental character of the work is borne in mind, a remarkable testimony to the welcome it has received. From the point of view of the Pure Mathematician, in the modern sense of the term, the publication of the new volume is an event. Few indeed are the Pure Mathematicians of to-day who can afford to neglect the study of Dr. Hobson's book, and even the most profoundly read will certainly find in its pages much that will be new to them, and much that will stimulate them to renewed research.

The fundamental position held by the Theory of Functions of the Real Variable in modern mathematical science is now all but universally recognised, and the progress that has been made in the theory is rightly regarded as one of the very greatest achievements of the twentieth century.

The first volume of the second edition appears at a critical moment in the history of human thought and effort. Two rival principles are face to face ; on the one hand it is urged that all human effort should be devoted to practical purposes, and that the study only of those branches of human knowledge should be seriously pursued which may result in a larger measure of material progress. To those who hold this view the line of least resistance is that which consists in ignoring the claims and depreciating the value of Pure Mathematical Research, and inevitably the theories which Dr. Hobson treats are even more vulnerable to this mode of attack than those with which less recent research has occupied itself. On the other hand, many of the best thinkers and the most experienced guiders of intellectual effort, whatever be the particular subject that they profess, philologists, historians, philosophers, Hellenists, lawyers, are agreed that the best preparation for accurate work, in these seemingly widely distant disciplines, is a preliminary training in exact mathematical reasoning.

The enemies of Pure Mathematics as an end in itself are not these, nor are they among the ranks of those who have most recently found themselves under the necessity of cultivating mathematics for the direct services it renders, geographers, economists, or indeed biologists and chemists. The danger comes from a certain section of the older school, including even some mathematicians in whose opinion, especially towards the end of the nineteenth century in England, mathematics was a rough tool to be employed as occasion might direct, and as the physicist and engineer might require. It is to such that the outside world repeatedly refers for a judgment of the value of mathematical researches, and donors are apt to ignore, or even expressly exclude, pure mathematical research in the terms of their gifts to learned societies and other foundations.

It is however as much a need of the mind to reason clearly as to know, and indeed knowledge which is not based on clear thinking is of little account. The danger is great lest, if a breach is made at the extreme wing, the whole intellectual position may be carried by assault. That the line is still in good formation is due, in fact, more to the prevailing ignorance of the aims of scientific workers than to the existence of any real differentiation, from the point of view of the practical man, between the work of natural philosophers and that of pure mathematicians.

A distinguished botanist had occasion recently to ask a town councillor for help in a fight against the suppression of a mathematical chair. It was to be a chair having for duties a course of Calculus and a course of Mechanics.

The councillor was mollified by the mention of Mechanics, because he said he realised that there were still machines to be invented, but he saw no occasion to have the Calculus taught. Such a man would make short work of most of our distinguished physicists should he learn, as some day he may, that their work has no more direct connection with the problems of electrical, mechanical and other industries than the theories treated of in the book before us. In the interests of science as a whole we hope that the intellectual world may see in the success of the present volume an earnest of the success that will be achieved if a united front be shown to the common enemy.

The volume before us has no fewer than 670 pages, and we may fairly assume that the second volume will have approximately the same length. The first edition contained 770 pages in all. This points to an increase of at least 500 pages, and will give the reader some idea of the labour involved in the compilation, labour which can have been little inferior to that required in the original composition. The progress of the science since 1907 has indeed been such as to render it impossible, even in this edition, to cover the whole ground, and this although some space has been saved by the adoption in many cases of a simpler phraseology and a happier choice of terms, such as the use of the words "bounded," "cell," "net," and the phrase "the fitting of systems of nets to a cell," etc.

The systematic use of these and similar terms must be regarded as one of the happy innovations of the second edition. In the same connexion we note that the word "integrable" has definitively displaced the word "summable," and even "totalisation" has been merged in "integration," while the older forms of integrals have taken their place as particular cases of the general concept. The attentive reader will find other improvements in the minutiae of the writing to which it is hardly necessary to call attention here. In one respect there is no change where the reviewer would have been glad to see it, the word "enumerable" is used throughout as expressing the characteristic of the potency of the natural numbers. The spoken word is easily confused with "innumerable," and seems to the reviewer in no way as suitable as the homely word "countable."

The first 255 pages are taken up with preliminary notions. Chapter I. deals with Real Numbers, and in Chapters II., III. and IV. the author gives an account of certain parts of the Theory of Sets of Points. It is not till Chapter V. that we enter on the main subject. We pass first over definitions of the terms in use and questions of continuity and discontinuity to Functions of Bounded Variation, Maxima and Minima, Differential Coefficients and Derivates—or, as the author prefers to call them, Derivatives—Implicit Functions, Space-filling Curves. The remaining three chapters of the volume are devoted to the subject of Integration: Chapter VI. The Riemann Integral; Chapter VII. The Lebesgue Integral; and Chapter VIII. Non-absolutely Convergent Integrals.

A notable feature of this edition as compared with the previous one is the prominence given to what is known as the Multiplicative Axiom or Principle of Zermelo. The author has been at considerable pains to point out where this axiom is used in the proofs of theorems, and has endeavoured to construct demonstrations independent of it whenever possible, thereby rendering great services from the point of view of the logician and philosopher.

In Dr. Hobson's treatment of the Theory of Sets of Points we remark, in the edition before us, the far greater prominence given to non-linear sets of points. Much of the theory is independent in point of fact of the number of dimensions; where this is the case it has been scrupulously pointed out, and the language employed is usually such as to apply equally for any number of dimensions. This remark applies not only to the Theory of Sets properly so called, but also to the Theory of Integration, of which the former theory forms the base in the treatment of integrals adopted in the volume before us.

It is evidently impossible to form a final opinion with regard to the author's treatment of the theory of integration as a whole, as a fundamental part of the presentation, that relating to series and sequences, is reserved for the forthcoming volume; but even a cursory glance shows the immense progress that has been made in this connexion since the first edition. Much of the theory treated of under this head has been developed, some of it by new writers,

since the appearance of the first edition, and the author has shown wide erudition and great skill in his presentation of this matter, and in his co-ordination of what has been taken from many sources into an organic whole. The work thus achieved by Dr. Hobson must be regarded as of great scientific value; in speaking of this it may, however, be remarked that had the author found it consistent with his mode of treatment, much simplicity would have been gained by the use of the method of monotone sequences in proving what he calls the "fundamental approximation theorem." That and analogous theorems are in fact an intuitive consequence of the method in question.

The volume contains an extremely lucid and able account of the very difficult and somewhat intricate theory of non-absolutely convergent integrals of the most general known types, in which it may be noted sequences are virtually employed. This is the more noteworthy as Dr. Hobson's continental rivals in their presentation of modern work have so far not got beyond the Lebesgue integral. Again, by a very detailed discussion of the Stieltjes integral and allied concepts, such as the integrals of Hellinger, Dr. Hobson has conferred a great boon on the student.

The main object of the writer would seem to have been throughout that his book should serve as a guide to the large and growing literature of the subject. And here we must insist on the great value of such a careful set of references as those given in the text, while adding that these references would have been enhanced in interest if in each case the date had been added; not in view of questions of priority, but as a clue to the history of mathematical thought. A fuller index also would have rendered greater service; the work of preparing this might have been allotted to a pupil or assistant, as used to be done in the old Germany before the war.

In a further notice the reviewer proposes to discuss the details of the book. W. H. Young.

Plane Analytic Geometry. By C. E. Love. Pp. xiii+306. 10s. 6d. net. 1923. (Macmillan Co.)

Plane and Solid Analytic Geometry. By W. F. Osgood and W. C. Graustein. Pp. xvii+614. 14s. net. 1923. (Macmillan Co.)

It is doubtful whether either of these books meets the need of any considerable number of English students. Both deal with elementary analytic geometry, first plane and then of three dimensions, and both assume a student who has at times to be very tenderly dealt with. Professor Love interrupts his work on the circle (p. 76) with investigation of the roots of a quadratic, and yet has already talked of infinite roots of an equation on page 65. He encourages the use of the words *easily*, *obviously*, *evidently*, and gives the same title and type to such facts as the diameter property of a conic and the mechanical way of recognizing the equation to a hyperbola with axes parallel to the axes of coordinates.

Professors Osgood and Graustein state that they have in mind "the needs of the future specialist in geometry, the analyst, the mathematical physicist and the engineer," but all these need to progress. When they have reached the stage of analytical geometry they do not require elementary directions in plotting curves, instruction to draw a neat figure, and they can as a rule solve simple equations. They are ready for general equations, and are not helped by continual approach through particular numerical cases. They do need help as to rigorous proof, and frankly thinking about a smooth bead on a flexible inelastic string may illustrate, but it does not prove the fact that the tangent to an ellipse makes equal angles with the focal distances. Nor does the consideration of steps taken by cows entering a yard by a gate, going to the river to drink, and then keeping on to a door.

The authors approach the conics *seriatim* in an unusual way; it is doubtful whether it is really helpful in the end. L. Ashcroft.

Consciousness, Life and the Fourth Dimension. By R. Eriksen. Pp. xx+213. 10s. 6d. net. 1923. (Gyldendal.)

The author of this book is a lecturer in philosophy in the University of Christiania. His object is to develop the theory of relativity in the domain

of philosophy and psychology. The greater part of his work is outside the scope of the mathematician, and no new light seems to be thrown upon Einstein's theory. However, some portions of the book appear to deal with ordinary dynamics. Their nature may be shown by a quotation. On p. 171 we read, after a discussion of the motion of a stone thrown into the air : *While force is revealed in attraction and is connected with " action in distance," energy is repulsive and connected with " contact-action."*

Atomic Structure and Spectral Lines. By A. Sommerfeld. Translated by H. L. Brose. Pp. xiii + 626. 32s. net. 1923. (Methuen.)

This is a translation of the third edition of Sommerfeld's well-known treatise, which is one of the most valuable books on mathematical physics ever written. Its main object is to give a full and systematic account of the applications of the quantum theory * to the properties of atoms as revealed by spectroscopic observations. The earlier chapters deal with the leading facts concerning electrons, cathode rays, X-rays, radio-activity, the photo-electric effect, atomic numbers, and isotopes. These are followed by a discussion of the experiments of Laue and the Braggs upon X-ray spectra and crystalline structure. The book then becomes more mathematical, giving Bohr's theory of the Balmer series, assuming the orbit of the electron to be circular, and Sommerfeld's extension to elliptic orbits. Later chapters show how Bohr's theory can be applied to a great variety of phenomena, including the Stark effect and the Zeeman effect (normal and anomalous). Finally we have Sommerfeld's brilliant combination of the theories of relativity and quanta, by which he accounted for the " fine structure " of the hydrogen lines.

There is no attempt to conceal the weak points of these new theories. " Both points of view, the classical continuous wave theory and the discontinuous statistical theory of light-quanta, each offer at present only one-half of the truth. How the dilemma will be overcome finally cannot yet be gauged. At all events the classical wave theory in its application to the phenomena of light propagation has not yet been supplanted by something better."

The book concludes with seventeen appendices. These are much more difficult than the preceding matter, which is fairly straightforward and elementary. One appendix applies contour integration to the evaluation of definite integrals. Others deal very fully with Hamiltonian mechanics. " Up to a few years ago it was possible to consider that the methods of mechanics of Hamilton and Jacobi could be dispensed with for physics, and to regard it as serving only the requirements of the calculus of astronomic perturbations and the interests of mathematics. Accordingly it is not even touched on in the most famous of German text-books on mechanics, namely the lectures of Kirchhoff. Since the appearance of the papers by Schwarzschild and Epstein in the year 1916, which, following immediately on the papers by the author on the fine structure of the Balmer series, link up the quantum conditions with the partial differential equations of mechanics, it seems almost as if Hamilton's method were expressly created for treating the most important problems of physical mechanics."

To translate a book of this length must have been a heavy task, and it is not surprising that some errors have crept in. Unfortunately a few of these are rather misleading to a beginner, such as the rendering of *Äquivalentladung* by *electrochemical equivalent* (p. 4), or the insertion of a symbol in a place where it alters the meaning of a sentence and appears to assert a new physical law (line 6 of p. 45). However, the translator has rendered such a boon to English students by making this standard work more accessible to them that they will forgive his occasional lapses. The text and diagrams are clearly printed, though the price is rather high for a treatise which every mathematical physicist should possess.

H. T. H. Piaggio.

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* A fairly full account of the leading features of this theory is given in a review of Reiche's *Quantum Theory* (*Gazette*, March, 1923).

A Primer of Geometry. By W. PARKINSON, M.A., and A. J. PRESSLAND, M.A. 4s. 6d. Pt. I., 2s. 6d. Pt. II., 2s. 6d. 1923. (Clarendon Press.)

There are signs that the anarchy which followed the dethronement of Euclid and which is not to be altogether regretted is to be replaced by some sort of settled government whose constitution is founded on the experience of good and evil gained during the period of license. We may expect—we may even hope—that the text-books which survive the ordeal of continuous trial in the class-room will not be of one rigidly uniform type. Neither the "conservative" nor the "advanced" teacher should have difficulty in finding a suitable manual for class-work. The book before us would be an excellent choice for the former, retaining as it does the substantial merits of the "Elements" while avoiding their very serious defects as a class-book for beginners, and supplying valuable addenda. It is based on the recommendations of the Report on the Teaching of Geometry issued by the A.M.A.

The nine sections into which it is divided treat as follows: I. Of the fundamental geometrical concepts (Solid, Surface, Line, Point Dimension, Direction Angle), the nature of the last two being supposed led up to by a class discussion for which a suggestive set of questions is provided; II. Of fundamental facts concerning Angles, including the properties of Parallels; III. and IV. Of Triangles and Parallelograms; V. Of Areas; VI. Of Loci; VII. Of the Circle; VIII. Of Rectangles (with illustrations of algebraical identities); IX. Of Ratio and Similar Figures. Upwards of 900 exercises are supplied in nine corresponding sets. A welcome sketch of the history of Greek geometry and a useful collection of papers actually set by various examining bodies are added. The type and arrangement are good, the diagrams large and excellent. Among the many signs of the experienced teacher we note especially (i) the remarks on drawing diagrams, (ii) the introduction to Loci. We do not think the treatment of parallels by "direction" quite up to the level of the rest of the book. We have no objection to allowing beginners to pass rapidly by the logical assumptions made, but, granted that the stage at which these difficult matters confront the learner is unsuitable for their discussion, it seems advisable that his attention should be called to them in a subsequent note. We should like to have seen "Perigal's Dissection" inserted among the exercises on the Theorem of Pythagoras.

["Dethronement" is milder than "Euclid dead and buried." It may not be out of place to remind our readers that Dodgson's "Euclid and his Modern Rivals" was not the first voice crying in the wilderness:

Euclid's/Elements/of/Geometry/from the/*Latin* Translation of COMMANDINE/. To which is added,/A Treatise of the *Nature* of *Arithmetic* of/Logarithms; Likewise/Another of the ELEMENTS of *Plain* and *Spherical*/Trigonometry; With/A PREFACE, shewing the *Usefulness* and Excellency of/this WORK.

By Doctor John Keil, F.R.S. and late/Professor of Astronomy in Oxford. The *Whole* Revised; where deficient, Supplied; where lost/or corrupted, Restored. Also/Many Faults committed by Dr. Harris, Mr. Caswel/Mr. Heynes, and other Trigonometrical/Writers, are shewn; and in those/Cases where *They* are mistaken, here are/given Solutions Geometrically True./An ample Account of which may be seen in the Preface,/by SAMUEL CUNN./The Fourth Edition./To which is subjoined an APPENDIX, containing the/Investigation of those Series's omitted by the Author,/And the Difference between Dr. Keil and Mr./Cunn impartially examined and adjusted.

London :/Printed for Tho. WOOD/WARD, at the *Half-Moon*, be/tween the *Two Temple-Gates* in *Fleet Street*; and sold by/J. Osborn, at the *Golden Ball* in Pater-noster-Row./MDCCXLI.

DR. KEIL'S PREFACE.

A young Mathematician may be surprised, to see the old obsolete ELEMENTS of EUCLID appear afresh in Print; and that too after so many new Elements of Geometry, as have been lately publish'd; especially since those who gave us the Elements of Geometry, in a new manner, would have us believe they have detected a great many faults in EUCLID. These acute PHILOSOPHERS pretend to have discovered, that EUCLID's Definitions are not perspicuous enough; that his Demonstrations are scarcely evident; that his whole Elements

are ill-dispos'd ; and that they have found out innumerable Falsities in them, which had lain hid to their Times.

But by their Leave, I make bold to affirm, that they carp at EUCLID undeservedly : For his Definitions are distinct and clear, as being taken from first principles, and our most easy and simple Conceptions ; and his Demonstrations elegant, perspicuous and concise, carrying with them such Evidence, and so much strength of Reason, that I am easily induced to believe that the Obscurity SCIOLISTS so often accuse EUCLID with, is rather to be attributed to their own perplexed Ideas, than to the Demonstrations themselves. And however some may find Fault with the Dispositions and Order of his Elements, yet notwithstanding I do not find any Method, in all the Writings of this kind, more proper and easy for Learners than that of Euclid.

It is not my Business here to answer separately every one of these CAVILLERS ; but it will easily appear to any one, moderately versed in these Elements, that they rather shew their own Idleness, than any real faults in EUCLID : Nay, I dare venture to say, there is not one of these new Systems, wherein there are not more faults, nay, grosser Paralogisms, than they have been able even to imagine in EUCLID. (Note *Keil* used for Keill.)—*Editor.*]

Mensuration and Elementary Solid Geometry for Schools. By R. M. MILNE, M.A. Pp. x+206. 8s. 6d. 1923. (Cambridge University Press.)

This seems a sound and thorough treatise on the lines on which the subject is taught at the Royal Naval College, Dartmouth, and likely to be found useful at other institutions for professional training. We can specially commend the parts of it which deal with Solid Geometry, and agree with the author in the stress which he lays on getting pupils into the way of drawing solid figures "that look right." It was a complaint of De Morgan's that "the diagrams by which mathematical students (and even writers) represent their solid figures are generally so imperfect that it may be worth while to explain how in all cases of sufficient importance a good drawing may be made with very little trouble," and he devoted a special article in the P.C. (of which a résumé is given in vol. iii. p. 258 of the *Gazette*) to show how orthographic projection of a set of axes of reference could be easily practised. Mr. Milne uses in preference the method of Oblique Projection (sometimes called Oblique Parallel Perspective), which is probably more adapted to his immediate purpose. In addition to the ordinary contents of a text-book on Mensuration, we have chapters on the Geometry of the Sphere, on Rectangular Axes in two and three dimensions, and on Descriptive Geometry. The last of these is welcome, as the subject has been too much and too long ignored in mathematical teaching. Proofs are given of the theorems on which the practical calculations are based ; a specially neat one is that of the content of the Paraboloid of Revolution, of which the nature may be briefly indicated as follows : Let (x_1, y_1), (x_2, y_2) (h, k) be three points on the parabola $y^2=4ax$, such that $x_1+x_2=h$. Then $y_1^2+y_2^2=\frac{k^2}{h}(x_1+x_2)=k^2$. Hence the average section of the paraboloid generated by the revolution about Ox from (0, 0) to the ordinate k is $\pi k^2/2$.

An interesting addition to the methods of approximate quadrature due to Simpson and Weddle is that given as Dufton's Rule (see *Nature*, May 1920). Collections of well-chosen numerical exercises suitable for familiarising students with the metrical properties of geometrical figures are supplied throughout. Explanations and demonstrations are clearly given. Arrangement and type are good.

Mathematics and Physical Science in Classical Antiquity. Translated from the German of J. L. HEIBERG by D. C. MACGREGOR. Pp. 110. 2s. 6d. 1922. (Oxford University Press.)

This is the second of two volumes in a series ("Chapters in the History of Science") edited by Mr. Charles Singer, which is intended to complete gradually an outline of the history of science. The first was a companion volume by the general editor, which deals more fully with medical and biological aspects.*

* Mr. Singer's *Greek Medicine and Biology* is reviewed in *Science Progress* for October.

Such volumes were much wanted. We would add to the noble sentence of Bacon's which appears on each issue of the *Gazette*, "and to make themselves acquainted with its history." When Mr. Heppel read his paper to the Association in 1893 *On the Use of History in Teaching Mathematics*, a question arose as to what book was available for English teachers who agreed with his views. As far as we can recollect, the only one adduced was Prof. Baden Powell's excellent little book, long out of print, *Natural Philosophy, an Historical View of the Progress of the Physical and Mathematical Sciences*. The works to which Miss Barwell in her paper on the same subject in 1913 referred to as the chief authorities for her historical sketch were Branford's *Study of Mathematical Education* and D. E. Smith's *Teaching of Elementary Mathematics*, neither of which valuable works is mainly devoted to the historical development of Mathematics. Valuable additions have been made to the resources of the mathematical teacher in this department of late years, especially in the form of papers contributed to the *Gazette* by Sir T. Heath, Mr. Rouse Ball and others. We do not know, however, of any small work which occupies just the ground of Prof. Heiberg's. It has many merits, not the least of which is that it is eminently readable: in fact, while our immediate duty was to devote ourselves to the mathematicians, we have been continually drawn off from their lives and work to those of the physicians, among whom Themiso of Laodicea is of special interest. It is a model of orderly arrangement of a mass of interesting material and as different as it well could be from that "lean and barren thing," deprecated by Miss Barwell, "a skeleton outline to be got up for examination." Prof. Heiberg gives a Bibliography which Mr. Singer has extended. This should be useful to those who are induced by this small treatise to study more deeply any of the various departments of Greek Science. As to the need of such study, for mathematical teachers at any rate, we agree with the views expressed by Prof. W. P. Milne in his address to the Leeds Meeting on the "Training of the Mathematical Teacher." EDWARD M. LANGLEY.

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199. (Algebra) is to the Mathematics the same that Logic is to the ordinary Philosopher, and has therefore been called *Logistic*, and is become so common amongst us, because of its engaging Beauty. . . . that even Ladies of the highest Quality have been induced to learn it; the Dutchess of E—— has attained so great a Degree of Perfection, as well in Numbers as Geometry, that Persons who make the greatest Figure for Learning have earnestly sought for the Honour of her Conversation. An Instance so illustrious ought to banish all sorts of Diffidence, and excite those that have their Ease. Ozanam's *Cursus Mathematicus*. Now done into English. 1712 (Preface, Vol. I., p. v).

An Ordinate Problem is that which can be done but only one way.

An Inordinate Problem is that which can be done an infinite number of ways. Ozanam's *Cursus Mathematicus*, Vol. I., p. 3.

A Problem which is possible, but which has not ever been resolved, because of its seeming difficulty, is called an *Apote*; as is now (by some) the *Squaring the Circle*. . . . *ditto*, p. 4.

Algebra is a Science by means whereof we endeavour to resolve any possible Problem in the Mathematics, which is done by the means of a sort of literal Arithmetic, which for that reason has been call'd *Specious* Algebra, because its reasonings are all done by the species or forms of things, namely by the Letters of the alphabet. . . . *ditto*, p. 9.

200. "Besides, if $\frac{4}{\frac{2}{3}}$ were a fraction, the denominator $\frac{2}{3}$ would indicate that some unit had been divided into $\frac{2}{3}$ equal parts! Since it is impossible for *man* to so divide a unit, this species of complex fractions must be regarded as a special gift from on high, "for with God all things are possible." [From an article by the Principal of a school, in the . . . 1894.]

THE LIBRARY,

160 Castle Hill, Reading.

LOSSES. 1889-1923.

The following books cannot now be traced, although there is evidence that the Library has at some time possessed them. The titles are printed in the hope of catching the eye either of borrowers who have forgotten the loans or of benefactors who will make good the losses. Lest would-be donors should be discouraged, it should be said that this list shows the leakage of thirty-four years, during a great part of which time the books were accessible to every visitor in a room used by a succession of different societies not one of which was directly interested in the integrity of our collection. Under the present arrangements it is hardly possible for a book to disappear.

G. Boole	Finite Differences - - - - - - - - - -	1860
A. A. Bourne	Notes on Elementary Dynamics. *Given to A.I.G.T. by Author,* 1890.	
E. J. Brooksmith	Arithmetic for Beginners. *Given to A.I.G.T. by Publishers,* 1890.	
H. F. W. Cowley	Simple Interest. *Given to A.I.G.T. by Author,* 1890.	
C. L. Dodgson	Euclid I.-II. *Given to A.I.G.T. by Dr. T. A. Hirst,* 1890.	
A. G. Greenhill	Report on Gyroscope Theory. *Last recorded as returned by a borrower, Aug.* 1919.	
P. J. Harding	Geometry of Thales.	
E. W. Hobson	Mathematics from the Point of View of the Mathematician and the Physicist. *Last recorded as lent in April* 1918.	
J. Hymers	Conic Sections.	
D. Kikuchi	Plane Geometry. *English translation by the Author, given by him to A.I.G.T.,* 1891. *The Japanese original is still in its place, but as was said at the General Meeting when this gift was reported, "Possibly there are members who do not understand Japanese."*	
D. B. Mair	School Course of Mathematics - - - - - -	1907
J. J. Milne	Companion to Weekly Problem Papers. *Given to A.I.G.T. in* 1889 *by the Author, who has recently made good the loss. If there is a guilty borrower, perhaps he will relieve his conscience by offering some other book.*	
Oliver, Waite, and Jones	Algebra. *Given to A.I.G.T. by Prof. G. W. Jones,* 1889.	
J. Ozanam	Mathematical Recreations. Hutton's Edition. Vol. I. -	1803
	An odd volume; the edition was in four.	
G. Salmon	Conics. Fourth Edition - - - - - - -	1863
	The least serious of the losses, since we have the second edition and the sixth. But see note on Milne's "Companion".	

This is a convenient occasion to record the following losses; these books will not appear in future catalogues unless they are replaced:

A. R. Forsyth	Differential Equations - - - - - - - - -	1895
	Returned, April 1914, *rat-eaten too badly for use or repair. This is the only case on record in which a member has deliberately sent a book back in a damaged state.*	
A. C. Jones	Notes on Analytical Geometry.	
J. W. Russell	Pure Geometry. Sequel to Elementary Treatise. *The last three books were despatched by a borrower in Ireland, Nov.* 1920, *but failed to reach this country.*	

ADDITIONS.

The Librarian reports the following gifts:

From Mr. J. P. Clatworthy:

A collection of back numbers of the *Gazette*.

From Miss H. P. Hudson:

F. S. Woods	Higher Geometry	1922

From Prof. E. H. Neville:

P. Frost	Newton's Principia I.-III.	1863
J. A. Galbraith and S. Haughton	Manual of Hydrostatics	1859
P. M. H. Laurent	Traité d'Analyse. Vol. 3: Intégrales Définies et Indéfinies	1888
R. Woodhouse	Astronomy. Vol. 2 *Vol. 1 is already in the Library.*	1818

From Mr. C. W. Payne:

I. Barrow	Archimedis Opera, Apollonii Conica, et Theodosii Sphaerica	1675
S. H. Winter	Geometrical Drawing. Part II. *Filling one of the gaps to which attention was drawn in the October issue.*	1880

From Mr. F. Robbins:

A copy of No. 1 of the *Gazette*.

From Mr. A. Robson:

S. L. Loney	Trilinear Coordinates *The well-known text-book to which this is a sequel is not in the Library.*	1923

From Mr. R. S. Williamson:

The Teachers' Edition of his Unconventional Arithmetical Examples for Juniors - 1923

GENERAL TEACHING COMMITTEE.

Examination Questions Sub-Committee.—The members of the Sub-Committee appointed to consider criticisms of examinations are:—Miss E. Glauert, Girls' High School, Scarborough; A. W. Siddons, Rendalls, Harrow-on-the-Hill; W. J. Dobbs, 12 Colinette Road, Putney, S.W. 15.

Examination questions on pure or applied mathematics within the school range, if open to criticism, should be sent with comments to one of the above.

W. J. Dobbs, *Hon. Sec.*

NOTICES.

If possible, the January *Gazette* and the Geometry Report will be sent out together to members before Christmas.

The Editor will be grateful if the member to whom he sent, on its publication, a copy of Dr. Watson's *Bessel Functions* will kindly inform him of its whereabouts.

ERRATUM.

P. 377, under Fig. 2, *for* "Brigg" *read* "Briggs."

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
AT THE UNIVERSITY PRESS, GLASGOW.

BOOKS RECEIVED, CONTENTS OF JOURNALS, ETC.

December, 1923.

Constructive Arithmetical Exercises. Part II. Based on A. E. LAYNG'S Arithmetic (Extended with Reference Notes). By R. W. M. GIBBS. Pp. vii + 245-482 + 76. 3s. 6d. net. 1923. (Blackie.)

Practical Mathematics. By V. T. SAUNDERS. Pp. 46. 1s. 6d. 1923. (Bell.)

Atomic Structure and Spectral Lines. By A. SOMMERFELD. Pp. xiii + 626. 32s. net. 1923. (Methuen.)

Theory of Determinants. Vol. IV. By SIR THOMAS MUIR, C.M.G., F.R.S. Pp. xxxi + 508. 40s. net. 1923. (Macmillan.)

Biomathematics, being the Principles of Mathematics for Students of Biological Science. By W. W. FELDMAN. Pp. xix + 398. 21s. net. 1923. (Griffin.)

The Elements of Coordinate Geometry. Part II. *Trilinear Coordinates,* etc. By S. L. LONEY. Pp. viii + 228. 6s. 1923. (Macmillan.)

The Name "Divergent" Series. By F. CAJORI. P. 1. Reprint from *Bull. Amer. Math., Soc.* Feb., 1923.

Recent Symbolisms for Decimal Fractions. By F. CAJORI. Reprinted from *The Mathematics Teacher.* Pp. 183-187. March, 1923.

Sur les Forces Centrales. By B. HOSTINSKY. Pp. 11. 19 cts. 1922. *Sur les Equations aux Différences Finies qui sont identiques à leurs adjointes.* By JOS. KAUCKÝ. Pp. 3-12. 22 cts. 1922. *Remarques sur une Surface Cubique à Point Uniplanaire.* By L. SEIFERT. Pp. 3-11. 23 cts. 1923. *Sur la Réduction des Equations Différentielles à des Equations Intégrales.* By JOS. KAUCKÝ. Pp. 3-14. 24 cts. 1923. *Sur une Equation Singulière de Volterra de Première Espèce.* By JOS. KAUCKÝ. Pp. 3-16. 17 cts. 1922. (All Publications de la Faculté des Sciences de l'Université Masaryk.)

Academia pro Interlingua. Edited by Prof. G. PEANO. June, 1923.

Philip E. B. Jourdain (1879-1919). By LAURA JOURDAIN and GEORGE SARTON. Pp. 126-136. Reprint from *Isis.* Oct. 1922.

The Cycloid in Astronomy. By J. G. THOMSON. Pp. 1-14. 1s. April, 1923. (Printed by Allen, Elder St., Edinburgh).

A Primer of Geometry. By W. PARKINSON and A. J. PRESSLAND. Pp. 264. 4s. 6d. net; or Part I. Pp. 136. 2s. 6d. net. Part II. Pp. 129-264. 2s. 6d. net. (Clarendon Press). 1923.

Shop Mathematics. By J. M. CHRISTMAN. Pp. x + 316 + lxviii. 10s. net. 1923. (Macmillan Co.)

Carènes de Formes nuisibles ou favorables à leurs grandes vitesses et résistances de l'eau à leur translation. By F. E. FOURNIER. Pp. 27. 3 fr. 50. 1923. (Gauthier-Villars).

The Development of Mathematics in Bohemia. By QUIDO VETTER. Translated and edited with notes by Prof. R. C. ARCHIBALD. Reprint from *Amer. Math. Monthly.* Feb. 2, 1923. Pp. 47-58.

Mechanical Engineering Formulae. By E. H. HUDDY. Pp. 167. 4s. 6d. net. 1923. (Spon & Co.)

Chance and Error. The Theory of Evolution. By MARSH HOPKINS. Pp. iii + 223. 7s. 6d. net. 1923. (Kegan Paul.)

Consciousness, Life and the Fourth Dimension. By R. ERIKSEN. Pp. xx + 213. 10s. 6d. net. 1923. (Gyldendal.)

Relativity and Modern Physics. By G. D. BIRKHOFF with the co-operation of R. E. LANYER. Pp. xi + 283. 18s. 6d. net. 1923. (Harvard Univ. Press and Humphry Milford.)

An Introduction to Projective Geometry. By R. M. WINGER. Pp. xiii + 443. 12s. 6d. net. 1923. (D. C. Heath.)

Mathematical Signs of Equality. By F. Cajori. Pp. 116-125. Reprinted from *Isis*, Oct. 1922.

Plane and Solid Analytic Geometry. By W. F. Osgood and W. C. Graustein. Pp. xvii + 614. 14s. net. 1923. (Macmillan Co.)

Analytical Geometry. By C. E. Love. Pp. xiii + 306. 10s. 6d. net. 1923. (Macmillan Co.)

The New Physics. Lectures for Laymen and Others. By A. Haas. Trans. by R. W. Lawson. Pp. ix + 165. 6s. net 1923. (Methuen.)

Algebra for Schools. Part I. By J. Milne and J. W. Robertson. Pp. vii + 173 + xxxv. 2s. 6d. ; 3s. with Answers. 1923. (Bell.)

The Properties of Matter. By B. C. M'Ewen. Pp. vii + 316. 10s. 6d. net. 1923. (Longmans.)

Varieties of Minus Signs. By Prof. F. Cajori. Pp. 295-301. Reprint from *The Mathematics Teacher.* May 1923.

Science and Civilization. Essays arranged and edited by F. S. Marvin. (Unity Series. VI.) Pp. 350. 12s. 6d. 1923. (Ox. Univ. Press.)

Mensuration and Elementary Solid Geometry for Schools. By R. M. Milne. Pp. x + 206. 8s. 6d. 1923. (Cam. Univ. Press.)

Relativity. A Systematic Treatment of Einstein's Theory. By J. Rice. Pp. xv + 397. 18s. net. 1923. (Longmans.)

Elementary Mathematical Astronomy. By C. W. C. Barlow and G. H. Bryan. Pp. xvi + 445. 9s. 6d. net. 1923. (Univ. Tutorial Press.)

Les Fonctions Circulaires et les Fonctions Hyperboliques, étudiées parallèlement en partant de la definition géométrique. By H. Tripier. Pp. 58. 5 fr. 1923. (Vuibert, Paris.)

From Determinant to Tensor. By W. F. Sheppard. Pp. 127. 8s. 6d. net. 1923. (Clarendon Press.)

Grafting of the Theory of Limits on the Calculus of Leibniz. By F. Cajori. Pp. 223-234. Reprint from *The American Mathematical Monthly.* July-Aug. 1923.

Cours Complet de Mathématiques Spéciales. Tome IV. *Géométrie Descriptive et Trigonométrie.* By J. Haag. Pp. xi + 152. 13 fr. 1923. (Gauthier-Villars.)

Exercices du Cours de Mathématiques Spéciales. Tome IV. By J. Haag. Pp. 151. 13 fr. 1923. (Gauthier-Villars.)

Theorie der Gruppen von Endlicher Ordnung. By A. Speiser. Pp. viii + 194. 1.70$. 1923. (Springer, Berlin.)

Relativity and Gravitation. By T. P. Nunn. Pp. 162. 6s. net. 1923. (Univ. of London Press.)

Cours de Mathématiques Générales. Ie Partie. Calcul Différentiel. Géométrie Analytique à Deux Dimensions. By G. Verriest. Pp. 338. 38 fr. 1923. (Gauthier-Villars.)

A Short Course in Interpolation. By E. T. Whittaker and G. Robinson. Pp. viii + 71. 5s. net. 1923. (Blackie.)

The New Examiner. By P. B. Ballard. Pp. 269. 6s. net. 1923. (Hodder & Stoughton.)

The Principles of Relativity. By A. Einstein, H. A. Lorentz, H. Minkowski, A. Sommerfeld, and H. Weyl. Translated by G. B. Jeffery and W. Perrett. Pp. viii + 216. 12s. 6d. net. 1923. (Methuen.)

The Properties of Matter. By B. C. M'Ewen. Pp. vi + 316. 10s. 6d. net. 1923. (Longmans.)

Relativity, A Systematic Treatment of Einstein's Theory. By J. Rice. 18s. net. Pp. xv + 397. 1923. (Longmans.)

Modern Electrical Theory. Supplementary Chapters. Chap. XVII. *The Structure of the Atom.* By N. R. Campbell. Pp. ix + 161. 10s. net. (Cambridge Univ. Press.)

Clerk Maxwell's Electromagnetic Theory. By H. A. Lorentz. (Rede Lecture, 1923.) Pp. 35. 1s. 6d. net. 1923. (Cam. Univ. Press.)

The American Mathematical Monthly. (Lancaster, Pa.)

July-Aug. 1923.

Grafting of the Theory of Limits on the Calculus of Leibniz. Pp. 223-234. F. CAJORI. *The Four-Colour Problem.* Pp. 234-243. H. R. BRAHANA. *A Graphical Treatment of Mixtures.* Pp. 243-249. F. E. WOOD. *On the Circles of Antisimilitude of the circles determined by four given points.* Pp. 250-252. R. A. JOHNSON. *Should Book Reviews be censored?* Pp. 252-255. L. E. DICKSON. *Infinite and Imaginary Elements in Algebra and Geometry.* Pp. 255-256. T. FORT. *A Theorem concerning the Concurrency of four Planes.* Pp. 256-258. *On the Approximate Evaluation of the Complete Elliptic Integral.* P. 258. M. F. EGAN.

Sept.-Oct. 1923.

Vectorial Treatment of the Motion of a Rigid Body in a Plane. Pp. 290-296. E. L. REES. *Note on the Reliability of a Test.* [" Reliability " is the correlation between the results obtained by administering two forms of the same test in succession to the same group of candidates.] Pp. 296-301. W. L. CRUM. *Similitudinous and Anti-similitudinous Transformations in a Plane.* Pp. 301-306. W. H. ECHOLS. *The Method of Moments.* Pp. 307-311. D. JACKSON. *Another Definition of Amicable Numbers.* [Three numbers form an amicable triple if each equals the sum of the proper divisors of the other two.] Pp. 311-315. B. F. YANNEY. *Actual Solution of Simultaneous Linear Numerical Equations.* Pp. 316-318. W. R. RANSOM. *The Ten most important Mathematical Books in the World.* Pp. 318-321. W. C. EELLS. *Note on the Algebraic Solution of the Cubic.* Pp. 321-323. E. J. OGLESBY. *The " Limiting Case " in Enumerative Geometry.* Pp. 323-324. A. A. BENNET.

Annals of Mathematics. (Lancaster, Pa., U.S.A.)

Dec. 1922.

Spherical Representation of Conjugate Systems and Asymptotic Lines. Pp. 89-98. W. C. GRAUSTEIN. *On a Short Method of Least Squares.* Pp. 99-108. B. H. CAMP. *On the Convergence of the Sturm-Liouville Series.* Pp. 109-120. J. L. WALSH. *The Functional Equation* $g(x^2)=2ax+[g(x)]^2$. Pp. 121-140. J. H. M. WEDDERBURN. *On Convergence Factors in Triple Series and the Triple Fourier's Series.* Pp. 141-166. B. M. EVERSULL. *On the Minimising of a Class of Definite Integrals.* Pp. 167-174. P. R. RIDER. *A Pythagorean Functional Equation.* Pp. 175-180. E. HILLE.

Bollettino della Unione Matematica Italiana. (Zanichelli, Bologna.)

Oct. 1923.

Applicazioni geometriche di una formola di F. Siacci. Pp. 121-125. GINO LORIA. *Considerazioni sulla nota formula idrodinamica di Daniele Bernoulli.* Pp. 125-128. U. CISOTTI. *Condizione necessaria e sufficiente affinchè due superficie (reali e regulari) a curvatura non costante siano suscettibili di una stessa metrica.* Pp. 128-132. U. CRUDELI. *Correlazioni geodetiche fra due superficie.* Pp. 132-134. A. TERRACINI. *Sul teorema di Clairaut.* Pp. 134-139. P. DORE.

Bulletin of the American Mathematical Society. (Lancaster, Pa.)

July, 1923.

Report on Continuous Curves from the View-point of Analysis Situs. Pp. 289-302. R. L. MOORE. *A Generalisation of a Property of an Acnodal Cubic Curve.* Pp. 303-308. H. HILTON. *On Uniform Limitedness of Sets of Functional Operations.* Pp. 309-315. T. H. HILDEBRANDT.

Oct. 1923.

On the Riemann Zeta Function. Pp. 337-340. C. F. CRAIG. *Note on a certain type of Ruled Surface.* Pp. 341-344. W. C. GRAUSTEIN. *The Differentiation of a Function of a Function.* Pp. 344. H. S. CARSLAW. *On the Relative Curvature of two Curves in V.* Pp. 345-348. J. LIPKA. *Square-Partition Congruences.* Pp. 349-355. E. T. BELL. *Determination of all Systems of ∞^4 curves in space in which the sum of the angles of every triangle is two right angles.* Pp. 356-366. J. DOUGLAS.

Bulletin of the Calcutta Mathematical Society. (Calcutta Univ. Press.)

June, 1923.

On the Steady Translation and Revolution of a Liquid Sphere with a Solid Core. Pp. 1-12. S. C. MITRA. *Higher Order Tides in Canals of Variable Section.* Pp. 19-24. N. N. SEN. *On the Product of Bessel Functions.* Pp. 25-30. K. BASU. *Longitudinal Vibrations of a Hollow Cylinder.* Pp. 31-40. J. GHOSH. *A General Theorem for the Representation of X, where X represents the Polynomial* $\frac{x^p-1}{x-1}$. Pp. 41-54. *On the Covariant Curves of a Singular n-ic.* Pp. 55-64. B. S. MADHAVARAO.

Gazeta Matematica. (Chibrituri, Bucharest.)

Aug. 1923.

Generalizarea unei probleme de Aritmetica. Pp. 441-447. GH. BUICLIU.

Intermédiaire des Mathématiciens. (Gauthier-Villars.)

March-April, 1923.

The Journal of the Indian Mathematical Society. (Varadachari, Madras.)

April, 1923.

Operational Methods in Heat Conduction (cont.). Pp. 25-26. T. S. NARAYANA. *Time, Space, Matter and Mind.* Pp. 27-33. S. V. RAMAMURTHI. *Miquel Theorems.* Pp. 34-39. V. RAMASWAMI AIYAR. *The Parametric Representation of the Twisted Cubic.* Pp. 40-48. R. VAIDYANATHASWAMY. *Notes and Questions on a Theorem in Higher Plane Curves.* [On a polar line of any point A there are three points, the first polars of which have a point of inflection at that point.] Pp. 17-19. M. LAKSHMANUMURTI.

June, 1923.

Parametric Representations of the Twisted Cubic (cont.). Pp. 49-52. R. VAIDYAMTHASWAMY. *On the Equivalence of Three Fundamental Definitions of Irrational Number.* Pp. 53-62. B. S. MADHAVARAO. *On the Converse of Fermat's Theorem.* Pp. 62-68. N. B. MITRA. *Diffraction by a Rectangular Aperture.* Pp. 69-72. S. S. RAU. Notes:—*Normals from a point to a Quadric in n Dimensions.* Pp. 33-35. *Note on a Formula in Solid Geometry.* Pp. 36-39. E. H. NEVILLE.

Periodico di Matematiche. (Zanichelli, Bologna.)

July, 1923.

Come si è affermata la nozione della forma ellissoidica della Terra. Pp. 277-291. G. CASSINIS. *Poliedri regolari, semiregolari, e simmetrici.* Pp. 292-308. A. MARONI. *Calcolo approssimato delle radici di numeri reali nel sistema decimal.* Pp. 308-314. A. LA BARBERA. *Lo sviluppo d'una matrice quadrata di 4° ordine.* Pp. 314-316. G. POATO. *Notizia storica su la Regola di Torricelli per la misura dei volumi.* Pp. 317-319. E. BORTOLOTTI.

Revista de Matemáticas y Fisicas Elementales. (Calle Perú 147, ix. Buenos Aires.)

July, 1923.

Calculo rápido de determinantes. Pp. 49-55. E. REBUELTO. *Los Logaritmos y la división abreviada.* Pp. 55-58. B. IG. BAIDAFF.

Scientia. (Williams & Norgate.)

I. IX. 1923.

Origine e primo inizio del calcolo degli immaginari. Pp. 385-394. E. BORTOLETTI.

I. X. 1923.

The Consistency of the Non-Euclidean Geometries and the Impossibility of proving the Parallel Postulate. Pp. 73-82. H. S. CARSLAW.

School Science and Mathematics. (Smith & Turton, Mount Morris, Illinois.)

April, 1923.

Tangent Straight Lines among the Greeks. Pp. 320-322. G. A. MILLER. *Reorganisation of Secondary Mathematics.* Pp. 339-347. J. M. KINNY. *Mathematical Training as an Aid in removing Limitations.* Pp. 347-352. *The Teaching of Constructive Problems in Plane Geometry.* Pp. 353-360. B. M. VAUGHAN.

May, 1923.

Collegiate Courses in Mathematics for Prospective High School Teachers. Pp. 482-488. M. O. TRIPP.

December, 1940

THE LIBRARY.

THE COPSE, SONNING-ON-THAMES.

THE Librarian reports gifts as follows :

From Prof. **T. A. A. Broadbent:**

American Mathematical Society Semicentennial Publications (2 vols.) - - - - - - - 1938
I: R. C. Archibald, *History of the Society*. II: Semicentennial Addresses.

From Mr. **A. Eagle,** offprints of three papers.

From Mr. **W. L. Ferrar:**

G. VERRIEST Cours de Mathématiques Générales ; I - - - - 1923
T. 2 was already in the Library.

From Dr. **G. J. Lidstone:**

P. S. DE LAPLACE Œuvres ; I-VI - - - - - - - - 1843-1846
Traité de Mécanique Céleste (5 vols.), *Exposition du Système du Monde*. This edition is of treatises only, and was completed in 1848 by the *Théorie analytique des Probabilités*.

J. H. POINCARÉ Méthodes Nouvelles de la Mécanique Céleste ; II, III - - - - - - - - - 1893, 1899
Has any member the first volume to give?

From Mr. **H. J. Woodall:**

APOLLONIUS OF PERGA
Locorum Planorum Libri Duo - - - - - 1749
Restoration by R. Simson.

From Dr. **F. A. Yeldham:**

D. E. SMITH Addenda to *Rara Arithmetica* - - - - - 1939

From the Royal Society of S. Africa, parts of their *Transactions* containing papers by T. Muir missing from the Association's collection.

From the Editor of *Mathematical Reviews*, in exchange for current numbers of the *Gazette* :

A. A. ALBERT Structure of Algebras - - - - - - - - 1939
American Mathematical Society Colloquium 24.

From Miss **A. M. Anderson,** Miss **R. A. Clayton,** and Mr. **P. D. Goodall,** collections of *Gazettes*.

The following has been bought :

L. CARROLL A Tangled Tale - - - - - - - - - - 1885
A book of puzzles.

SCHOOL SCIENCE AND MATHEMATICS.

Several small gaps have been filled recently, and the run of this American periodical is now complete from Vol. 9, 1909, to date.

UNIVERSIDAD NACIONAL DE LA PLATA.

One number of the *Contribuciones al Estudio de las Ciencias Fisicas y Matematicas* has been missing from our set for many years ; it is out of print, but the author, Dr. J. R. Castiñeiras, has sent us a copy, and thanks to his courteous generosity our run of the *Contribuciones* is now perfect.

QUEENSLAND BRANCH.

REPORT FOR 1939–1940.

THE Annual Meeting was held at the University on 24th May, 1940, at which the Eighteenth Annual Report was presented.

The previous annual meeting was held on 21st April, 1939, and the Annual Report and the Financial Statement for the year then closing were presented and were adopted. The officers for the coming year were elected. The Presidential Address by Professor Simonds was on the subject " Newton and Leibnitz ".

During the year three General Meetings were held at the University. At the first, on 9th June, 1939, a discussion on " The Teaching of Mechanics " was opened by Mr. E. W. Jones, B.A.; at the second, on 4th August, Mr. J. P. McCarthy, M.A., read a paper on " Some curves and their history ", and at the third, on 27th October, Miss E. H. Raybould, M.A., read a paper on " Napier and his logarithms ".

The number of members of the Society at present is 23, of whom 11 are members of the Mathematical Association. The Financial Statement shows a credit balance of £7 17s. Copies of *The Mathematical Gazette* come to hand regularly and are circulated as usual among members of the Branch who are not members of the M.A. So far the publication of the *Gazette* has not been affected very much by the war. J. P. McCarthy, Hon. Sec.

SHEFFIELD BRANCH.

For the period of the war the Branch has adopted the policy of holding meetings during the summer months so as to obviate the difficulties of " black-out " travelling. Three meetings have been held this year and all have been keenly attended.

The Annual General Meeting was held on Tuesday, 23rd April, when officials for the current year were elected. Our new President is Mr. L. Smith, Headmaster of the Central Secondary School for Boys, Sheffield. The business meeting was followed by a lecture on " Some solved and unsolved problems in the theory of numbers ", given by Dr. R. Rado of the University of Sheffield.

The second meeting was held on 4th June, when Mr. R. M. Gabriel of Leeds University gave a talk entitled " Atomism and Continuity from the Greeks to Modern Times ".

At the final meeting held on 24th September we welcomed Mr. C. W. Tregenza, H.M.I., who dealt with the " General Organisation of School Mathematical Teaching ". The lecturer brought up a number of controversial points which were discussed by the members.

BOOKS RECEIVED FOR REVIEW.

S. Chapman and **J. Bartels**. *Geomagnetism. I. Geomagnetic and related phenomena.* Pp. xxviii, 542. *II. Analysis and physical interpretation of the phenomena.* Pp. x, 543-1049. 63s. 1940. (Oxford)

J. L. Coolidge. *A history of geometrical methods.* Pp. xviii, 451. 30s. 1940. (Oxford)

L. Crosland. *Upper school algebra.* Being an abridged and revised edition of Hall and Knight's *Higher algebra.* Pp. xv, 292. With or without answers, 6s. 1940. (Macmillan)

H. Dingle. *The special theory of relativity.* Pp. vii, 94. 3s. 6d. 1940. Monographs on physical subjects. (Methuen)

E. R. Greenhood. *A detailed proof of the chi-square test of goodness of fit.* Pp. xii, 61. 8s. 1940. (Harvard University Press; Oxford University Press)

D. Humphrey. *Revision mathematics for School Certificate.* Pp. viii, 315. 4s. 1940. (Longmans)

THE MATHEMATICAL ASSOCIATION.

(*An Association of Teachers and Students of Elementary Mathematics.*)

"*I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.*"—BACON.

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Editor of *The Mathematical Gazette*:

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THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and has exerted an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Southampton, Bangor, Yorkshire, Bristol, Manchester, Cardiff, Sydney (New South Wales), and Queensland (Brisbane). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"**The Mathematical Gazette**" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) SHORT NOTICES of books not specially dealt with in the REVIEWS;

(5) QUERIES AND ANSWERS, on mathematical topics of a general character.